Fully developed anelastic convection in a plane layer

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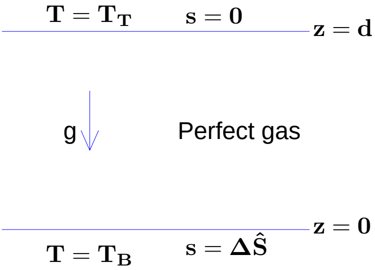
Anelastic plane layer convection



Plane layer, gravity vertical, non-rotating.

No-slip, fixed entropy boundaries.

Perfect gas $p = R\rho T$, p pressure, ρ density and T temperature.

Anelastic approximation is valid when entropy drop across layer is small,

$$\frac{\Delta \hat{S}}{c_p} \ll 1,$$

c_p is specific heat at constant pressure, assumed constant.

In stars, anelastic approximation often valid in the deep interior, but not near the surface, where convection is inefficient.

Entropy relations

Entropy for a perfect gas is

$$s = c_v \ln \left[\frac{p}{\rho^\gamma} \right] - s_0$$

$\gamma = c_p/c_v$ ratio of specific heats: c_v specific heat at constant volume. Assume $\gamma = 5/3$. Differential form:

$$ds = c_v \frac{dp}{p} - c_p \frac{d\rho}{\rho},$$

so if ds is small,

$$\frac{dp}{p} \approx \frac{\gamma d\rho}{\rho}$$

This is the adiabatic relation, so the layer is close to adiabatic. The adiabatic reference state is the solution of

$$\frac{d\bar{p}}{dz} = -g\bar{\rho}, \quad \bar{p} = R\bar{\rho}\bar{T}, \quad \frac{d\bar{p}}{d\bar{\rho}} = \frac{\gamma\bar{p}}{\bar{\rho}}$$

The adiabatic reference state

The solution of these equations is a polytrope

$$\overline{T} = T_B \left(1 - \tfrac{\theta z}{d}\right), \quad \overline{\rho} = \rho_B \left(1 - \tfrac{\theta z}{d}\right)^{3/2}, \quad \bar{p} = p_B \left(1 - \tfrac{\theta z}{d}\right)^{5/2}.$$

where d is layer depth, T_T and T_B top and bottom temperatures, subscript B being value at bottom and subscript T top.
$z = 0$ is bottom of layer, $z = d$ top of layer. Constants are defined by

$$\frac{gd}{c_p} = \Delta T = T_B - T_T > 0, \quad \theta = \frac{\Delta T}{T_B}, \quad \Gamma = \frac{T_B}{T_T} = \frac{1}{1-\theta}.$$

$\Gamma = \frac{T_B}{T_T}$ is a measure of how compressible the layer is.
Boussinesq is $\Gamma \to 1$, Γ large is very compressible. The number of density scale heights (defined at the top of the layer) that fit into the layer is $N_\rho = 1.5(\Gamma - 1)$.

The adiabatic reference state has constant entropy, so it doesn't satisfy $s_B = \Delta\hat{S}$, $s_T = 0$. The conduction state, which is different from the adiabatic reference state, is the solution of

$$\frac{d\tilde{p}}{dz} = -g\tilde{\rho}, \quad \tilde{p} = R\tilde{\rho}\tilde{T}, \quad \nabla^2\tilde{T} = 0,$$

the third equation being different. The solution is also a polytrope, but with a slightly different polytropic index.

$$\tilde{T} = T_B\left(1 - \tilde{\theta}\frac{z}{d}\right), \;\; \tilde{\rho} = \rho_B\left(1 - \tilde{\theta}\frac{z}{d}\right)^{\tilde{m}}, \;\; \tilde{p} = \frac{gd\rho_B}{\tilde{\theta}\,(\tilde{m}+1)}\left(1 - \tilde{\theta}\frac{z}{d}\right)^{\tilde{m}+1},$$

$$\widetilde{\Delta T} = T_B - \tilde{T}_T > 0, \quad \tilde{\theta} = \frac{\widetilde{\Delta T}}{T_B}, \quad \tilde{m} = \frac{gd}{R\widetilde{\Delta T}} - 1.$$

In this conduction state we take T_B to be the same as before, but $\tilde{T}_T$ isn't necessarily the same as T_T.

The conduction state entropy

The small anelastic parameter ϵ is now defined as

$$\epsilon = \tilde{\theta}\frac{\tilde{m}+1-\gamma\tilde{m}}{\gamma} = -\frac{d}{T_B}\left[\left(\frac{\mathrm{d}\tilde{T}}{\mathrm{d}z}\right)_B + \frac{g}{c_p}\right] \ll 1,$$

and the entropy in the conduction state is

$$\epsilon\tilde{s} = c_v \ln\frac{\tilde{p}}{\tilde{\rho}^{\gamma}} + \mathrm{const} = \frac{\epsilon c_p}{\theta}\ln\left(1-\theta\frac{z}{d}\right) + \mathrm{const},$$

correct to $O(\epsilon)$, and $\tilde{m} = m - 2.5\epsilon/\theta$. Since the boundaries have fixed entropy, the entropy at the boundaries in the conduction state defines the entropy drop across the layer for all Rayleigh numbers, so

$$\Delta\hat{S} = \frac{\epsilon c_p}{\theta}\ln\Gamma = \epsilon\Delta S,$$

so ϵ is determined by the small entropy drop across the boundaries.

Anelastic equations

The anelastic equations are derived by setting the full thermodynamic hatted quantities to

$$\hat{p} = \bar{p} + \epsilon p, \quad \hat{\rho} = \bar{\rho} + \epsilon \rho, \quad \hat{T} = \bar{T} + \epsilon T,$$

with

$$u \sim (\epsilon g d)^{1/2}, \quad t \sim \left(\frac{\epsilon g}{d}\right)^{-1/2}, \quad \hat{s} = \epsilon s,$$

putting them into the full equations of motion, subtracting off the reference state variables and ignoring any $O(\epsilon^2)$ terms.

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} = -\nabla\left(\frac{p}{\bar{\varrho}}\right) + \frac{g}{c_p} s \hat{\mathbf{e}}_z + \frac{\mu}{\bar{\rho}}\left[\nabla^2 \mathbf{u} + \frac{1}{3}\nabla\,(\nabla \cdot \mathbf{u})\right],$$

$$\nabla \cdot (\bar{\varrho}\mathbf{u}) = 0,$$

$$\bar{\rho}\bar{T}\left[\frac{\partial s}{\partial t} + \mathbf{u} \cdot \nabla s\right] = k\nabla^2 T + \mu\left[q + \partial_j u_i \partial_i u_j - (\nabla \cdot \mathbf{u})^2\right],$$

$$\frac{p}{\bar{p}} = \frac{\rho}{\bar{\rho}} + \frac{T}{\bar{T}}, \qquad s = c_v \frac{p}{\bar{p}} - c_p \frac{\rho}{\bar{\rho}}, \quad c_p - c_v = R.$$

Boussinesq, Anelastic, Fully compressible?

Here the dissipation

$$q = (\partial_j u_i)(\partial_j u_i) + \frac{1}{3}(\nabla \cdot \mathbf{u})^2 .$$

Boussinesq equations: $\Gamma \to 1$ limit of the anelastic equations.

$\bar{\rho}$ and $\bar{T}$ become constants, entropy s/c_p is replaced by temperature term αT.

$\nabla \cdot \bar{\rho}\mathbf{u} = 0$ replaced by $\nabla \cdot \mathbf{u} = 0$. The viscous heating terms in the energy equation are negligible.

Fully compressible equations have $\mathbf{u}$ the same order as the sound speed $\sim (gd)^{1/2}$, whereas in anelastic convection sound waves are removed because $\mathbf{u} \sim (\epsilon gd)^{1/2}$.
$\partial\rho/\partial t$ back in mass conservation equation. Timesteps have to be small enough to resolve sound waves. Toroidal-poloidal expansion no longer useful.

Alternative formulations for anelastic convection

When solved numerically, the full thermodynamic variables can be found by adding ϵT, $\epsilon\rho$ etc. to the reference state values.

Decoupled entropy formulation: In a turbulent regime, $k\nabla^2 T$ can reasonably be replaced by $\nabla \cdot \bar{\rho}\bar{T}\kappa_s \nabla s$, and this decouples the entropy and the temperature, a significant simplification. Not done here, because boundary layers may be laminar. Constant entropy boundaries required for full decoupling.

Constant entropy boundaries used here: $\hat{T} = \bar{T} + \epsilon T$, and T won't be zero on the boundaries. To leading order, $\bar{T}$ has the reference state value, but departures ϵT from the reference state won't satisfy $T = 0$ on the boundaries.

With constant temperature boundary conditions, $T = 0$ would be imposed, but then entropy would fluctuate on the boundaries.

Adiabatic reference state or conduction reference state?

Some authors use the adiabatic state as the reference state, so $\hat{T} = \bar{T} + \epsilon T$, but some use the conduction state as the reference state.

With the conduction state, $\hat{T} = \tilde{T} + \epsilon T$, etc.

Does it matter? Both give the same answers for the hatted variables, but the answers for T, ρ etc are different in the two cases, because $\hat{T}$ and $\tilde{T}$ differ at $O(\epsilon)$.

Need to be very careful comparing results from the different approaches! The corrections are however easy to work out and only depend on z.

Total heat flux carried by conduction is $-kd\hat{T}/dz = -kd\bar{T}/dz - k\epsilon dT/dz$ but we ignore the constant heat flux conducted down the adiabat.

In spherical geometry the divergence of the conducted flux is non-zero, and then the conducted flux can be a heat source/sink for the convection.

We call T the superadiabatic temperature, and define the conducted part of the superadiabatic heat flux

$$\epsilon F_{cond}^{super} = -\epsilon k \frac{dT}{dz}\bigg|_{z=0}, \quad \text{so} \quad F_{cond}^{super} = -k\frac{dT}{dz}.$$

At the boundaries, all the heat is carried by conduction, so

$$F^{super} = -k\frac{dT}{dz}\bigg|_{z=0} = -k\frac{dT}{dz}\bigg|_{z=d}.$$

The heat conducted down the non-convecting conduction state is

$$\epsilon F_{cond} = k\left[-\frac{\mathrm{d}\tilde{T}}{\mathrm{d}z} - \frac{g}{c_p}\right] = \frac{\epsilon k T_b}{d}.$$

Nusselt and Rayleigh number in anelastic convection

Nusselt number in anelastic convection defined as the ratio of the superadiabatic heat flux to the heat conducted down the conduction state superadiabatic gradient,

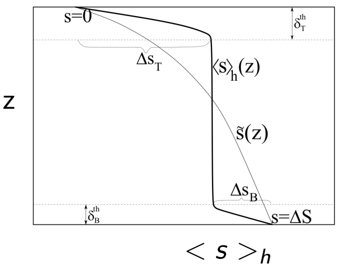
$$Nu = \frac{F^{super}}{F_{cond}} = \frac{F^{super} d}{k T_B},$$

so Nu is close to unity near onset, and is large in fully developed convection. For fixed entropy boundary conditions, the Rayleigh number is defined as

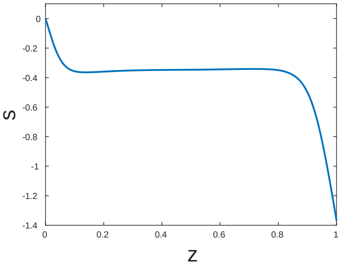
$$Ra = \frac{g \Delta S d^3 \rho_B^2}{\mu k} = \frac{c_p \Delta S \Delta T d^2 \rho_B^2}{\mu k}.$$

Note that small superadiabatic temperature gradient does not imply small Rayleigh number Ra, since the diffusion coefficients can be small, in fact to get $Ra \sim O(1)$ in the limit $\epsilon \to 0$ we must have

$$\frac{k}{\rho_B c_p} \sim \left(g d^3 \epsilon\right)^{1/2}, \quad \frac{\mu}{\rho_B} \sim \left(g d^3 \epsilon\right)^{1/2}.$$



Schematic entropy profile



Numerical entropy profile

Entropy is the conserved quantity with small diffusion and small dissipation, so it is the well-mixed quantity (not temperature).

Large Rayleigh number, so expect well-mixed turbulent flow in the interior, with asymptotically thin boundary layers near the boundaries. Horizontally averaged entropy $< s >_h$ shown.

Mouloud Kessar's numerical solution uses entropy diffusion, but otherwise similar. $Ra = 10^6$, $Pr = 1$, $\Gamma = 1.94$, $\rho_B/\rho_T = 2.71$.

The thickness of the top and bottom thermal boundary layers is δ_i^{th}, with $i = B$ or $i = T$. In the Boussinesq limit they are the same, but in anelastic convection they are different. Jumps in entropy across the layers are ΔS_i. Clearly

$$\Delta S_B + \Delta S_T = \Delta S.$$

Not much vertical motion in boundary layers, so expect hydrostatic boundary layers

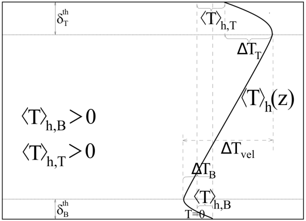
$$(\Delta p)_i \approx \frac{g}{c_p} \rho_i \Delta s_i \delta_i^{th},$$

which leads to

$$\frac{(\Delta T)_i}{T_i} \approx \frac{(\Delta s)_i}{c_p} \left(1 + \theta \frac{\delta_i^{th}}{d} \frac{T_B}{T_i} \right).$$

For thin boundary layers (incompressible b.l.) $\frac{(\Delta T)_i}{T_i} \approx \frac{(\Delta s)_i}{c_p}$.

Superadiabatic temperature profile



Horizontally averaged superadiabatic temperature $<T(z)>_h$.

$$< u_z >_h = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}z}\left\langle \bar{\rho} u_z^2 \right\rangle_h + \frac{\mathrm{d}\left\langle \hat{p}\right\rangle_h}{\mathrm{d}z} = -g\left\langle \hat{\rho}\right\rangle,$$

$$\text{constant entropy interior} \Rightarrow \hat{p} = K\hat{\rho}^{\gamma}, \quad \hat{p} = R\hat{\rho}\hat{T},$$

$$\frac{d<T>_h}{dz} = -\frac{1}{c_p\hat{\rho}}\frac{d}{dz} < \bar{\rho}u_z^2 >, \;\; \Delta T_{vel} = \int_{\delta_B^{th}}^{d-\delta_T^{th}} < \bar{\rho}u_z^2 > \frac{d}{dz}\left(\frac{1}{c_p\bar{\rho}}\right) dz > 0.$$

So temperature slope must be subadiabatic in the interior.

Energy balance integral

Multiply Navier-Stokes by $\bar{\rho}\mathbf{u}$ and integrate over the whole volume,

$$\frac{g}{c_p}||\bar{\rho}u_z s|| = \mu||q||, \quad q = (\partial_j u_i)(\partial_j u_i) + \frac{1}{3}(\nabla\cdot\mathbf{u})^2 .$$

where $||\ ||$ is volume average, assuming statistically stationary state. Integrating energy equation from 0 to z

$$\begin{aligned} F^{super} &= -k\frac{dT}{dz}\Big|_{z=0} = \left\langle\bar{\rho}\bar{T}u_z s\right\rangle_h - k\frac{dT}{dz} + \frac{g}{c_p}\int_0^z \left\langle\bar{\rho}u_z s\right\rangle_h \mathrm{d}z \\ &- \mu\int_0^z \langle q\rangle \,\mathrm{d}z - 2\mu\left[\left\langle u_z\frac{du_z}{\mathrm{d}z}\right\rangle_h - \frac{m\Delta T}{\bar{T}d}\left\langle u_z^2\right\rangle_h\right]. \end{aligned}$$

Letting $z\to d$ we get flux in equals flux out,

$$F^{super} = -k\frac{dT}{dz}\Big|_{z=0} = -k\frac{dT}{dz}\Big|_{z=d},$$

For incompressible b.l.s we deduce

$$F^{super} \sim \frac{kT_B\Delta S_B}{\delta_B^{th}c_p} = \frac{kT_T\Delta S_T}{\delta_T^{th}c_p}.$$

Entropy balance integral

Divide energy equation by $\bar{T}$ and integrate from 0 to z.

$$\langle \bar{\rho} u_z s \rangle_h = -\frac{k}{T_B}\frac{\mathrm{d}}{\mathrm{dz}}\langle T \rangle_h \Big|_{z=0} + \frac{k}{\bar{T}}\frac{\mathrm{d}}{\mathrm{dz}}\langle T \rangle_h \Big|_z + \int_0^z \frac{\mu}{\bar{T}}\langle q \rangle_h \, dz$$

$$-\int_0^z \frac{k\Delta T}{\bar{T}^2 d}\frac{\mathrm{d}}{\mathrm{dz}}\langle T \rangle_h \, dz + \int_0^z \frac{\mu}{\bar{T}}\langle \partial_j(\partial_i(u_i u_j)) - 2\partial_j(u_j(\partial_i u_i))\rangle_h \, dz.$$

Terms in red are small when the boundary layers are thin, and the viscous dissipation term is dominated by the boundary layer contributions. As $z \to d$ we get overall entropy balance

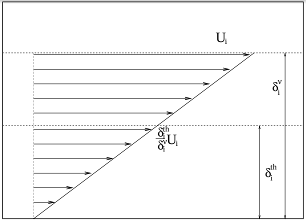
$$\frac{k}{T_B}\frac{\mathrm{d}}{\mathrm{dz}}\langle T \rangle_h \Big|_{z=0} - \frac{k}{T_T}\frac{\mathrm{d}}{\mathrm{dz}}\langle T \rangle_h \Big|_{z=d} \sim \int_0^d \frac{\mu}{\bar{T}}\langle q \rangle_h \, dz.$$

This gives the estimate from the boundary layers

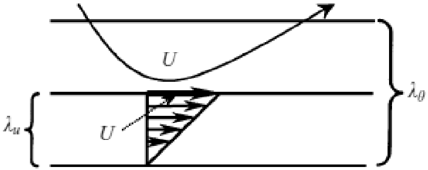
$$F^{super}\left(\frac{1}{T_T} - \frac{1}{T_B}\right) \sim \mu\left[\frac{U_B^2}{T_B \delta_B^\nu} + \frac{U_T^2}{T_T \delta_T^\nu}\right]$$

where U_B, U_T are horizontal velocities just outside the b.l.s.

Prandtl number effects: boundary layer nesting



$Pr > 1$



$Pr < 1$

$Pr = \mu c_p/k$. $Pr > 1$, thermal boundary layer inside viscous layer, $Pr < 1$ viscous layer inside thermal layer. U_i is large scale flow speed in bulk, δ_i^{ν}, δ_i^{th} viscous and thermal b.l. thicknesses.

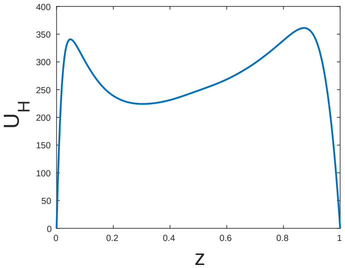
Large Pr: Velocity at edge of thermal b.l. $= \delta_i^{th} U_i/\delta_i^{\nu}$. Advection balances diffusion in the boundary layers,

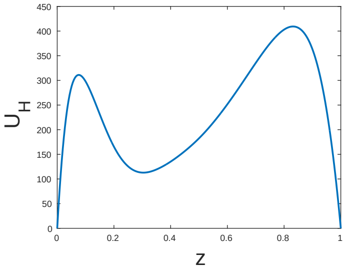
$$\frac{\rho_i U_i^2}{d} \sim \frac{\mu U_i}{(\delta_i^{\nu})^2}, \quad \frac{\delta_i^{th}}{\delta_i^{\nu}} \frac{\rho_i c_p U_i T}{d} \sim \frac{kT}{(\delta_i^{th})^2}, \quad \Rightarrow \frac{\delta_i^{\nu}}{\delta_i^{th}} \sim Pr^{1/3}.$$

Similarly, for low Pr,

$$\frac{\delta^{\nu_i}}{\delta_i^{th}} \sim Pr^{1/2}.$$

Horizontally averaged horizontal velocity profiles:





Pr=1, Ra=6×10^6, $\Gamma = 4.64$: Pr=10, Ra=3×10^6, $\Gamma = 4.64$

Grossmann and Lohse 2000 noted that in Boussinesq turbulent convection there are large scale components of the flow which maintain the viscous and thermal boundary layers.

Numerical solutions suggest this happens in anelastic high Rayleigh number convection, with well-defined U_B and U_T, but the ratio $r_u = U_T/U_B$ is no longer unity.

The velocity ratio

Multiply the equation of motion by $\bar{\rho}\mathbf{u}$ and horizontally average over the bulk interior, ignoring the small viscous term in the bulk, to get

$$\frac{1}{2}\frac{\partial}{\partial z}\left(\bar{\rho}\left\langle u_z u^2\right\rangle_h\right) \approx \frac{g}{c_p}\left\langle\bar{\rho}u_z s\right\rangle_h .$$

If viscous dissipation is mainly in boundary layers, entropy flux $\left\langle\bar{\rho}u_z s\right\rangle_h$ should be constant in the bulk, and all three components of velocity similar in the bulk. So just outside the boundary layers,

$$\frac{\partial}{\partial z}\left(\bar{\rho}\left\langle u_z u^2\right\rangle_h\right)|_T \approx \frac{\partial}{\partial z}\left(\bar{\rho}\left\langle u_z u^2\right\rangle_h\right)|_B .$$

The vertical scale heights in the bulk do vary with the pressure scale height, so $d/dz \sim 1/H_i$, giving

$$\frac{\rho_T u_T^3}{H_T} \sim \frac{\rho_B u_B^3}{H_B} \Rightarrow r_u^3 \sim \Gamma^{1/2} \Rightarrow r_u \sim \Gamma^{1/6}.$$

The boundary layer ratios

Three key ratios:

- Bulk horizontal velocity ratio $r_u = U_T/U_B$
- Boundary layer thickness ratio $r_\delta = \delta_T^{th}/\delta_B^{th} = \delta_T^\nu/\delta_B^\nu$
- Entropy jump ratio $r_s = \Delta S_T/\Delta S_B$

Energy balance gives

$$\frac{kT_B\Delta S_B}{\delta_B^{th} c_p} = \frac{kT_T\Delta S_T}{\delta_T^{th} c_p} \Rightarrow r_s = \Gamma r_\delta$$

Boundary layer balance gives

$$\frac{\rho_i U_i^2}{d} \sim \frac{\mu U_i}{(\delta_i^\nu)^2} \Rightarrow r_u r_\delta^2 \sim \frac{\rho_B}{\rho_T} = \Gamma^{3/2}$$

Determining the ratios

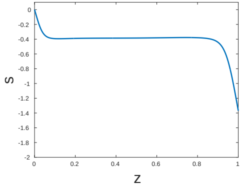
The three ratio equations are

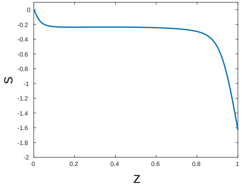
$$r_s = \Gamma r_\delta, \quad r_U {r_\delta}^2 = \Gamma^{3/2}, \quad r_u = \Gamma^{1/6},$$

giving

$$r_u = \Gamma^{1/6}, \quad r_\delta = \Gamma^{2/3}, \quad r_s = \Gamma^{5/3}, \quad r_T = \Gamma^{2/3}$$

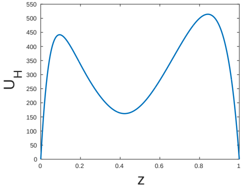
which fit the numerical data reasonably well.
e.g. at $Pr = 1$, $Ra = 3 \times 10^6$, $\Gamma = 1.94$, the numerics gets $r_u = 1.11$ (predicted 1.12), $r_\delta = 1.54$ (predicted 1.56), $r_s = 2.93$ (predicted 3.03)
Taking the horizontal boundary layer length scale varying with H_i gives $r_\delta = \Gamma^{1/6}$, which doesn't fit so well. Data is of course only at moderately high Ra.
Higher Γ gets the right trends, but the agreement between theory and numerics isn't quite as good. Maybe due to incompressible boundary layer assumption breaking down, or to difficulties with numerics.

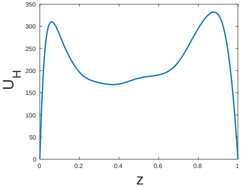




$\Gamma = 1.94,\ Ra = 3 \times 10^6,\ Pr = 1$: $\quad \Gamma = 2.92,\ Ra = 3 \times 10^6,\ Pr = 1$

Horizontally averaged entropy profiles. With $\Gamma = 2.92$, ratios $r_S = 5.90$ (predicted 5.97), $r_\delta = 2.13$ (predicted 2.04)





Horizontally averaged $\sqrt{u_x^2 + u_y^2}$. $1.94^{1/6} = 1.2,\ r_u = 1.1$:

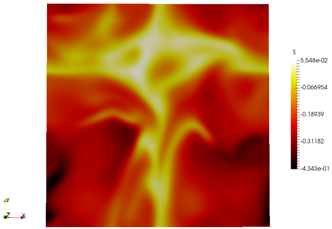
$2.92^{1/6} = 1.20,\ r_u = 1.07$.

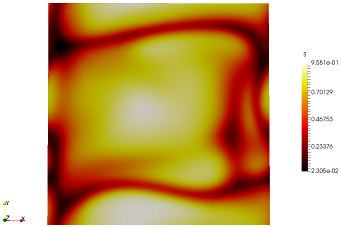
Conclusions from numerical results

- Reasonable support for the theoretical ratio estimates.

- Difficult to get long runs at *Ra* large enough for thin boundary layers, particularly at large Γ.

- For $\Gamma < 10$, r_δ results support horizontal length-scale *d* in the boundary layers (as in Boussinesq case), but for $\Gamma > 10$ r_δ grows less rapidly than $\Gamma^{2/3}$, suggesting a switch to a pressure scale height horizontal length scale.





Entropy slices at fixed z (Horizontal plane). Left $z = 0.1$ (bottom), right $z = 0.9$ (top). $Ra = 10^6$, $Pr = 0.7$, $\Gamma = 1.94$. Narrow sheet-like downdraughts near the top, more diffuse near bottom. Small scale structures occur near top, but so do large scale features, which may explain why at moderate Γ the horizontal length scales controlling the boundary layers are similar.

Nusselt number - Rayleigh number scalings

Equations needed

$$Nu = \frac{F^{super} d}{k T_B} \sim \frac{\Delta S_B d}{c_p \delta_B^{th}} = \frac{\Delta S d}{(1 + r_s) c_p \delta_B^{th}} = \frac{d}{\delta_B^{th}} \frac{\Gamma \ln \Gamma}{(1 + r_s)(\Gamma - 1)}$$

valid fror incompressible boundary layers.

$$Ra = \frac{c_p \Delta S \Delta T d^2 \rho_B^2}{\mu k}.$$

The entropy balance integral,

$$F^{super} \left(\frac{1}{T_T} - \frac{1}{T_B} \right) \sim \mu \left[\frac{U_B^2}{T_B \delta_B^\nu} + \frac{U_T^2}{T_T \delta_T^\nu} \right].$$

The viscous boundary layer equation, assuming moderate Γ

$$\bar{\rho} \left(\mathbf{u} \cdot \nabla \right) \mathbf{u}_h = \mu \nabla^2 \mathbf{u}_h \Rightarrow \rho_B \frac{U_B^2}{d} \sim \frac{\mu U_B}{(\delta_B^\nu)^2},$$

Prandtl number relation $\delta_B^\nu / \delta_B^{th} \sim Pr^{1/3}$ for large Pr, $Pr = \mu c_p / k$.

Nusselt number - Rayleigh number formula

Entropy balance becomes

$$\frac{\mu k^2 Ra}{c_p^2(1+r_s)T_T d^2 \rho_B^2 \delta_B^{th}} \sim \frac{\mu U_b^2}{T_B \delta_B^\nu}\left[1+\frac{\Gamma r_u^2}{r_\delta}\right].$$

Use the boundary layer balance to eliminate U_B in favour of δ_B^ν,

$$\frac{\mu k^2 Ra Pr^{1/3}}{c_p^2(1+r_s)T_T \rho_B^2 d^2} \sim \frac{\mu}{T_B}\left(\frac{\mu d}{\rho_B(\delta_B^\nu)^2}\right)^2\left[1+\frac{\Gamma r_u^2}{r_\delta}\right].$$

Now eliminate d/δ_B^ν in terms of Nu to get

$$Nu = Ra^{1/4} Pr^{-1/12}\frac{\Gamma \ln \Gamma}{\Gamma - 1}(1+r_s)^{-5/4}\left[1+\frac{\Gamma r_u^2}{r_\delta}\right]^{-1/4},$$

which is the scaling when $Pr \geq 1$, the boundary layers are incompressible and the dissipation is in the boundary layers. Mostly true in simulations.

Conclusions

• High *Ra* anelastic convection leads to a well mixed entropy zone bounded by thin layers.

• Assuming, like Grossmann and Lohse (2000), the boundary layers are controlled by advection from large scale horizontal flows just outside the boundary layers, we can derive scaling laws for *Nu* and the flow magnitude in terms of *Ra*. Entropy dissipation integral is crucial.

• Progress made on the ratio problem, but still work to do.

• Range of different scalings possible, depending on *Pr*, whether top b.l. is incompressible, and where the dominant dissipation lies.

• Agrees with the numerical simulations we have.