Fully developed anelastic convection with viscous dissipation in the bulk

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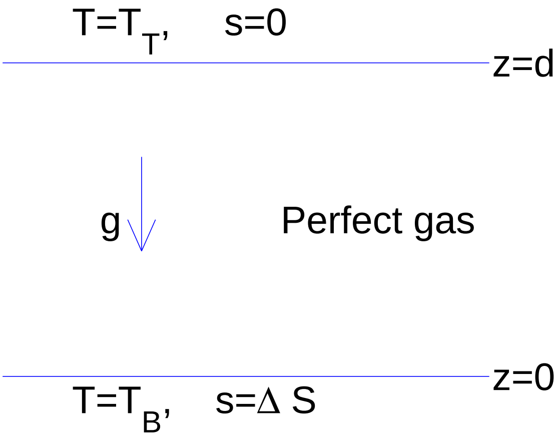
Formulation of the problem



Equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla\frac{p'}{\tilde{\rho}} + \frac{s'}{c_p}g\hat{\mathbf{e}}_z + \frac{\mu}{\tilde{\rho}}\nabla^2\mathbf{u} + \frac{\mu}{3\tilde{\rho}}\nabla(\nabla\cdot\mathbf{u}) \tag{1a}$$

$$\nabla\cdot(\tilde{\rho}\mathbf{u}) = 0, \tag{1b}$$

$$\tilde{\rho}\,\tilde{T}\left[\frac{\partial s'}{\partial t} + \mathbf{u}\cdot\nabla\left(\tilde{s}+s'\right)\right] = \nabla\cdot(k\nabla T) + 2\mu\mathbf{G}^s : \mathbf{G}^s - \frac{2}{3}\mu(\nabla\cdot\mathbf{u})^2, \tag{1c}$$

$$\frac{\rho'}{\tilde{\rho}} = -\frac{T'}{\tilde{T}} + \frac{p'}{\tilde{p}}, \qquad s' = -R\frac{p'}{\tilde{p}} + c_p\frac{T'}{\tilde{T}}, \tag{1d}$$

where we have assumed

$$\mu = \text{const}, \quad k = \text{const}, \quad g = \text{const}, \quad Q_{rad} = 0, \tag{2}$$

$$p = \rho R T, \qquad s = c_v \ln \frac{p}{\rho^\gamma}. \tag{3}$$

<u>DIFFUSIVE REFERENCE STATE:</u>

$$\tilde{T} = \tilde{T}_B \left(1 - \tilde{\theta}\frac{z}{d}\right), \qquad \tilde{\rho} = \tilde{\rho}_B \left(1 - \tilde{\theta}\frac{z}{d}\right)^m, \tag{4a}$$

$$\tilde{s} = c_p \frac{m + 1 - \gamma m}{\gamma} \ln\left(1 - \tilde{\theta}\frac{z}{d}\right) + \text{const}, \tag{4b}$$

where

$$m = \frac{gd}{R\widetilde{\Delta T}} - 1, \qquad \tilde{\theta} = \frac{\Delta\tilde{T}}{\tilde{T}_B}, \qquad \Delta\tilde{T} = \tilde{T}_B - \tilde{T}_T. \tag{5}$$

Notation

Let us introduce a stratification parameter

$$\tilde{\Gamma} = \frac{\tilde{T}_B}{\tilde{T}_T} = \frac{1}{1-\tilde{\theta}} > 1, \tag{6}$$

and a measure of small departure from adiabaticity

$$\varepsilon = \frac{d}{\tilde{T}_B}\left(\frac{\Delta \tilde{T}}{d} - \frac{g}{c_p}\right) > 0, \qquad \varepsilon \ll 1. \tag{7}$$

Directly from the state equation, we can calculate a relation between pressure, temperature and entropy jumps across boundary layers

$$\frac{(\triangle p')_i}{\tilde{p}_i} = \frac{\gamma}{\gamma - 1}\left[\frac{(\triangle T')_i}{\tilde{T}_i} - \frac{(\triangle s)_i}{c_p}\right] + \mathscr{O}\left(\varepsilon^2\right).$$

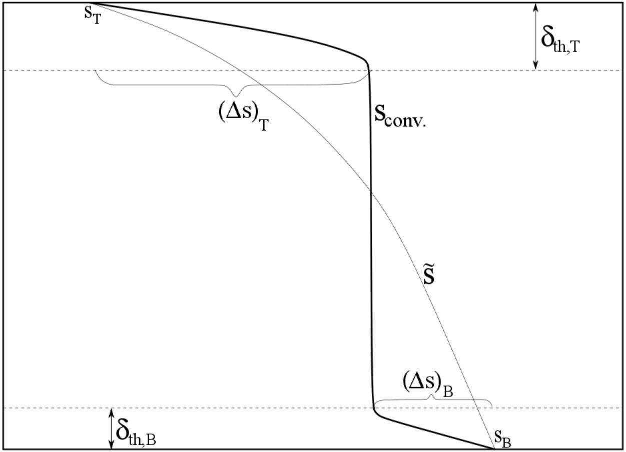
$(i = T\ (top)$ or $B\ (bottom))$.
The pressure jump is due to weak buoyancy in BL's, therefore

$$\frac{(\triangle T')_i}{\tilde{T}_i} \approx \frac{(\triangle s)_i}{c_p}\left(1 + \tilde{\theta}\frac{\delta_{th,i}}{d}\frac{\tilde{T}_B}{\tilde{T}_i}\right) + \mathscr{O}\left(\varepsilon^2\frac{\delta_{th,i}}{d}\right). \qquad (8)$$

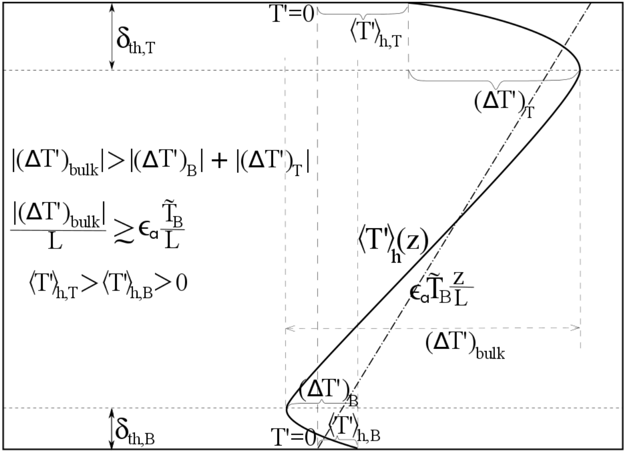
We assume, that startification not strong enough, to be visible in boundary layers

$$\tilde{\theta}\frac{\delta_{th,T}}{d}\frac{\tilde{T}_B}{\tilde{T}_T} = \left(\tilde{\Gamma} - 1\right)\frac{\delta_{th,T}}{d} \ll 1.$$

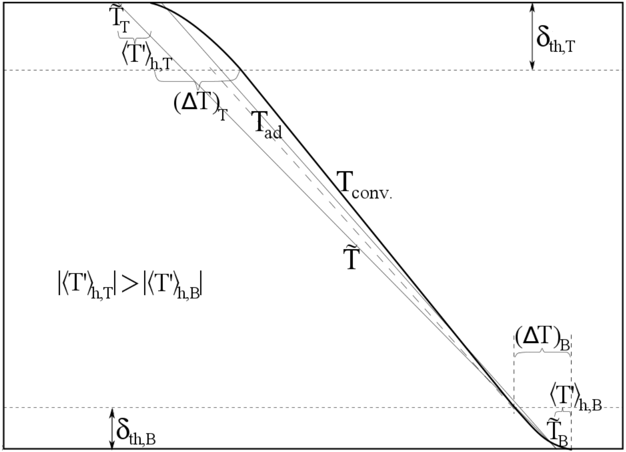
Vertical profile of entropy



Vertical profile of temperature fluctuation



Vertical profile of total temperature



Definitions

Definitions of the Nusselt, Rayleigh and Reynolds numbers

$$Nu = \frac{F_S(z=0)}{\tilde{F}_S} \approx \frac{(\triangle T')_B\, d}{\varepsilon\, \tilde{T}_B \delta_{th,B}}, \tag{9}$$

$$Ra = \frac{g \triangle \tilde{s} d^3 \tilde{\rho}_B^2}{\mu k} \approx \frac{c_p \triangle \tilde{s} \triangle \tilde{T} d^2 \tilde{\rho}_B^2}{\mu k}, \tag{10}$$

$$Re_B = \frac{U_B d \tilde{\rho}_B}{\mu}, \qquad Re_T = \frac{U_T d \tilde{\rho}_T}{\mu}. \tag{11}$$

Ratios

Furthermore, we define

$$r_s = \frac{(\Delta s)_T}{(\Delta s)_B} = \tilde{\Gamma} r_T, \qquad r_T = \frac{(\Delta T')_T}{(\Delta T')_B},$$

$$r_\delta = \frac{\delta_{th,T}}{\delta_{th,B}} = \frac{\delta_{\nu,T}}{\delta_{\nu,B}},$$

$$r_U = \frac{U_T}{U_B}$$

Boundary layer thickness

Because

$$(\Delta s)_B + (\Delta s)_T = (\Delta s)_B (1 + r_s) = \Delta \tilde{s} = c_p \varepsilon \frac{\tilde{\Gamma}}{\tilde{\Gamma} - 1} \ln \tilde{\Gamma},$$

we can relate $(\Delta T')_B$ to $\Delta \tilde{s}$ and from the definition of Nu obtain

$$\frac{\delta_{th,B}}{d} \approx \frac{\tilde{\Gamma} \ln \tilde{\Gamma}}{(1 + r_s)\left(\tilde{\Gamma} - 1\right)} Nu^{-1},$$

$$r_\delta \overset{\text{def.}}{=} \frac{\delta_{th,T}}{\delta_{th,B}} \approx \tilde{\Gamma}^{-1} r_s = r_T.$$

Estimates of ratios

Next we balance mean inertia against the work of the buoyancy

$$\frac{1}{2\tilde{\rho}}\frac{\partial}{\partial z}\left(\tilde{\rho}\left\langle u_z u^2\right\rangle_h\right) \approx \frac{g}{c_p}\left\langle u_z s'\right\rangle_h,$$

so that

$$\frac{c_p}{g}\frac{U_T^3}{D_{\rho,T}} \approx \left[\left\langle u_z s'\right\rangle_h\right]_T, \qquad \frac{c_p}{g}\frac{U_B^3}{D_{\rho,B}} \approx \left[\left\langle u_z s'\right\rangle_h\right]_B.$$

We need to be able to compare the advective flux at the top and bottom !!!

Estimates of superadiabatic heat flux

$$F_S(z=0) \approx -k\frac{\mathrm{d}}{\mathrm{d}z}\left(\tilde{T}+\langle T'\rangle_h - T_{ad}\right) + \tilde{\rho}\,\tilde{T}\,\langle u_z s'\rangle_h + \frac{\Delta\tilde{T}}{d}\int_0^z \tilde{\rho}\,\langle u_z s'\rangle_h \mathrm{d}z - \mu\int_0^z \langle q\rangle_h \mathrm{d}z,$$

$$F_S(z=0) \approx -k\frac{\tilde{T}_B}{\tilde{T}}\frac{\mathrm{d}}{\mathrm{d}z}\left(\tilde{T}+\langle T'\rangle_h - T_{ad}\right) + \tilde{\rho}\,\tilde{T}_B\,\langle u_z s'\rangle_h - \mu\int_0^z \frac{\tilde{T}_B}{\tilde{T}}\langle q\rangle_h \mathrm{d}z.$$

The second relation taken at $z=d$, by the use of $F_S(z=0)=F_S(z=d)$ leads to

$$F_S(z=0)\left(\frac{1}{\tilde{T}_T}-\frac{1}{\tilde{T}_B}\right) = \mu\int_0^d \frac{1}{\tilde{T}}\langle q\rangle_h \mathrm{d}z. \qquad (13)$$

Estimates of ratios (2) - VD in the bulk

$$F_S(z=0) \approx \tilde{\rho}_T \tilde{T}_T \left[\left\langle u_z s' \right\rangle_h\right]_T \approx \tilde{\rho}_B \tilde{T}_B \left[\left\langle u_z s' \right\rangle_h\right]_B, \tag{14}$$

since at the bottom the work of the buoyancy force and viscous dissipation integral are negligible and at the top, accroding to the global balance $g \langle \tilde{\rho} u_z s' \rangle / c_p = \mu \langle q \rangle$ they are approximately equal and thus cancel out. Consequently

$$\frac{[\langle u_z s' \rangle_h]_T}{[\langle u_z s' \rangle_h]_B} \approx \frac{\tilde{\rho}_B \tilde{T}_B}{\tilde{\rho}_T \tilde{T}_T} = \tilde{\Gamma}^{m+1} \approx \frac{U_T^3}{U_B^3} \frac{D_{\rho,B}}{D_{\rho,T}}, \tag{15}$$

thus

$$r_U = \tilde{\Gamma}^{m/3}, \tag{16}$$

$$r_\delta = r_T = \tilde{\Gamma}^{m/3}, \quad r_s = \tilde{\Gamma}^{m/3+1}. \tag{17}$$

Example m=3/2

$$r_U = \tilde{\Gamma}^{1/2}, \quad \text{as opposed to} \quad r_U = \tilde{\Gamma}^{1/6},$$

$$r_\delta = r_T = \tilde{\Gamma}^{1/2}, \quad \text{as opposed to} \quad r_\delta = r_T = \tilde{\Gamma}^{2/3},$$

$$r_s = \tilde{\Gamma}^{3/2} \quad \text{as opposed to} \quad r_s = \tilde{\Gamma}^{5/3}$$

Green estimates correspond to VD in boundary layers

Scaling laws (1)

$$\mu \int_0^d \langle q \rangle_h \mathrm{d}z \approx \mu \int_{\delta_{th,B}}^{d-\delta_{th,T}} \langle q \rangle_h \mathrm{d}z \approx \tilde{\rho}_B U_B^3 = \frac{\mu^3}{\tilde{\rho}_B^2 L^3} Re_B^3, \qquad (18)$$

since the viscous dissipation is dominant in the bulk, where it is expected to balance the nonlinear inertial term. Moreover, since $r_U^3 = \tilde{\Gamma}^m$ in the current case, the maximal estimate is obtained either by taking $\tilde{\rho}_B U_B^3$ or equivalently $\tilde{\rho}_T U_T^3 \approx \tilde{\rho}_B U_B^3$.

This leads to

$$Ra Nu Pr^{-2} \approx \frac{\tilde{\Gamma} \ln \tilde{\Gamma}}{\tilde{\Gamma} - 1} Re_B^3. \qquad (19)$$

and

$$\frac{\delta_{th,T}}{\delta_{th,B}} \approx \frac{r_s}{\tilde{\Gamma}} \approx \tilde{\Gamma}^{m/3} > 1. \qquad (20)$$

Scaling laws (2)

For

$$1 \ll \tilde{\Gamma} \ll (PrRa)^{1/(2m+6)},$$

We get

$$Nu \approx \frac{\ln \tilde{\Gamma}}{\Gamma^{(2m+6)/5}} Pr^{1/5} Ra^{1/5},$$

$$Re_B = \tilde{\Gamma}^{2m/3} Re_T \approx \tilde{\Gamma}^{-(2m+6)/15} Pr^{-3/5} Ra^{2/5},$$

and consequently

$$\frac{\delta_{th,B}}{L} \approx \tilde{\Gamma}^{(m+3)/15} Pr^{-1/5} Ra^{-1/5},$$

Scaling laws - comparison, m=3/2

$$Nu \approx \frac{\ln \tilde{\Gamma}}{\Gamma^{9/5}} Pr^{1/5} Ra^{1/5};$$

$$Nu \approx \frac{\ln \tilde{\Gamma}}{\tilde{\Gamma}^{2}} Pr^{1/8} Ra^{1/4},$$

$$Re_B = \tilde{\Gamma} Re_T \approx \tilde{\Gamma}^{-3/5} Pr^{-3/5} Ra^{2/5};$$

$$Re_B = \tilde{\Gamma}^{4/3} Re_T \approx \tilde{\Gamma}^{-2/3} Pr^{-3/4} Ra^{1/2},$$

Scaling laws - comparison, m=3/2

$$\frac{\delta_{th,B}}{d} \approx \tilde{\Gamma}^{(m+3)/15} Pr^{-1/5} Ra^{-1/5};$$

$$\frac{\delta_{th,B}}{d} \approx \tilde{\Gamma}^{(2m+1)/12} Pr^{-1/8} Ra^{-1/4},$$

$$\frac{\delta_{th,T}}{\delta_{th,B}} \approx \tilde{\Gamma}^{1/2} > 1;$$

$$\frac{\delta_{th,T}}{\delta_{th,B}} \approx \tilde{\Gamma}^{2/3} > 1,$$

Summary

- Total vertical heat flux varies with height due to work of the buoyancy and viscous heating.
- Thicknesses of boundary layers increase with $\tilde{\Gamma}$ and $\delta_{th,T} > \delta_{th,B}$.
- The velocities are greater at the top,
- The latter two imply, that the top boundary layer is more prone to instability and is more likely to become compressible.