A Hamiltonian approach to a hybrid quasi-hydrostatic / pseudo-incompressible fluid model.

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September 27, 2019

Overview

Making our sound-proof models more versatile?

- Sound waves are fast.
- We would like to filter them from our equations.
- Many 'sound-proof' models/approximations - Boussinesq, quasi-hydrostatic, anelastic, 'low mach'...
- Constructing an approximation that works well in across multiple scales is challenging
- *Having a better approximation across multiple-scales motivates our 'hybrid' approximation.*

Making our sound-proof models more robust?

- In constructing such approximated equation sets, have to be careful to preserve (energy) conservation properties.
- *This motivates a Hamiltonian approach.*

When are sound waves fast?

In a gravitationally stratified fluid, there are two kinds of waves

$$\omega_s^2 = c^2 k^2 \text{ and } \omega_g^2 = \frac{N^2 k_h^2}{k^2}$$

When is $\omega_s \gg \omega_g$?

- Small vertical scale (large k_z) - Boussinesq regime.

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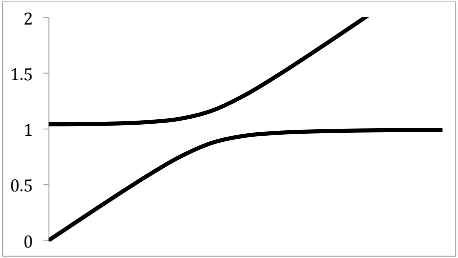
- Nearly adiabatic (small N) - anelastic regime.
- Small horizontal scale (large k_h) - pseudo-incompressible regime.
- Large horizontal scale (small k_h) - the quasi-hydrostatic regime.

No approximation works in all four limits.

Example: pseudo-incompressible versus quasi-hydrostatic approximation for an ideal, isothermal atmosphere

Linear perturbation analysis of an isothermal, hydrostatic ideal gas

$$0 = k_z^2 + \frac{1}{4}\left(\frac{N^2}{g} - \frac{g}{c^2}\right)^2 + \left(k_h^2 - \frac{\omega^2}{c^2}\right)\left(1 - \frac{N^2}{\omega^2}\right)$$



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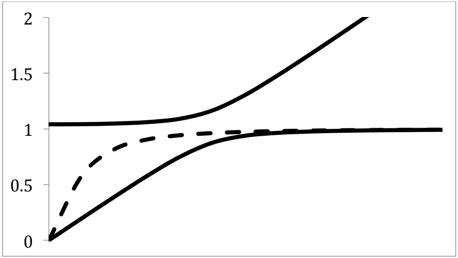

For $k_h \gg N/c$, $\omega^2 \approx \frac{N^2 k_h^2}{k^2 + \frac{1}{4}\left(\frac{N^2}{g} - \frac{g}{c^2}\right)^2}$

pseudo-incompressible

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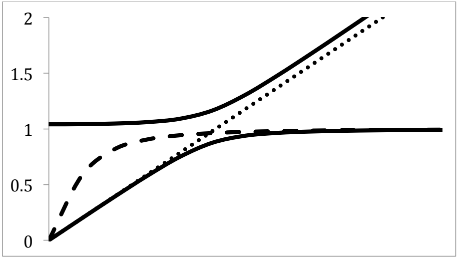

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pseudo-incompressible

For $k_h \ll N/c$, $\omega^2 \approx \frac{N^2 k_h^2}{k_z^2 + \frac{1}{4}\left(\frac{N^2}{g} + \frac{g}{c^2}\right)^2}$

quasi-hydrostatic

What is the pseudo-incompressible approximation, exactly?

$$\rho\left(p, S\right) \longrightarrow \rho\left(p_0, S\right)$$

- Fluid elements expand or contract instantaneously to achieve pressure equilibrium.
- *'Instantaneous acoustic equlibrization'*.
- Valid if pressure perturbation is small.

Pseudo-incompressible model - typical approach to derivation (truncated asymptotic expansion)

Characteristic scales are posited. Each dependent variable is then nondimensionalised and expanded as an asymptotic series in one or more small parameters (e.g. the Mach number).

Issues with energy conservation under typical approaches to a PI model

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- When additional physics needs to be included in the model (such as rotation or magnetic field), the derivation generally needs to be redone from scratch, and new scalings may be required.
- **There is no reason to expect a truncated asymptotic model to respect energy conservation that holds for the original, unapproximated system.**

Issues with energy conservation under typical approaches to a PI model

- Possible violation of conservation laws is the most serious, often leading to unphysical behaviour.
- E.g. some versions of analytic exhibit spurious wave growth[1], negative Rayleigh number convection[2],[3] and misrepresent the onset of the magnetobouyancy instability [4].
- Many asymptotically-derived models therefore reintroduce higher-order terms in a post-hoc fashion, in order to restore desired conservation.

[1] Brown et al. 2012 Astrophys. J. 756, 109.
[2] Calkins et al. 2015 Proc. R. Soc. Lond. A 471 (2175)
[3] Drew et al. 1995 Geo. Astro. FluidDyn. 80, 241.
[4] Wood et al. 2014 Phys. Plasmas 21 (5), 052110.

Resolving issues with energy conservation in PI models via a Hamiltonian approach

- Energy conservation (and other conservation laws) generally arise from the underlying Hamiltonian mathematical structure of the dynamical system (Noether's theorem).
- For a Hamiltonian system, all of the equations of motion can be derived from knowledge of a single quantity, $\mathcal{S}$ (*the action*) — by applying Hamilton's principle of stationary action, i.e. $\delta\mathcal{S} = 0$.
- Conservation laws will be 'built-in' from the start (at the expense of painful algebra / variational calculus).
- Non-ideal physics can be bolted on by demanding 'thermodynamic consistency'[5].

[5] Klein & Pauluis 2012 J. Atmos. Sci. 69, 961.

Resolving issues with energy conservation in PI models via a Hamiltonian approach

E.g. Fully compressible euler equations

An action of the form

$$\mathcal{S}\left[\mathbf{a}\right] = \iint \mathcal{L}\, \mathrm{d}^3\mathbf{x}\, \mathrm{d}t$$

with *Lagrangian density*

$$\mathcal{L} = \left[\frac{1}{2}\rho|\mathbf{u}|^2 - \Phi\rho - \rho U\right]$$

Applying Hamiltons principle can be shown to yield the usual compressible Euler equation of motion

- Rather than approximating the equations of motion directly, why not make approximations to the action *before* applying HP?
- The resulting equations are guaranteed to respect conservation.

A hybrid model

- We've seen that the large horizontal scales should be treated as quasi-hydrostatic. (PI elsewhere).
- There are a number of approaches that do so in the case of evolving background (e.g. MAESTROeX).
- ... but they don't necessarily have energy conservation built in.

[6] Vasil, G. M +, ApJ, 2013

[7] See also Klein & Pauluis 2012, arrives at same modified equations via asymptotics.

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- Why not impose two constraints simultaneously at the level of fluid action? $p = \bar{p}$ and $\partial \bar{p}/\partial z = -g\bar{\rho}$

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- Why not impose two constraints simultaneously at the level of fluid action? $p = \bar{p}$ and $\partial \bar{p} / \partial z = -g \bar{\rho}$
- Note, Hamiltonian approach to just the $p = \bar{p}$ constraint for non-evolving base has been considered [6], and MAESTROeX uses the resulting modifications by default.[7]

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A hybrid model

Applying two constraints at the level of fluid action.

Modify the Lagrangian density with the constraints

$$p = \bar{p} \text{ and } \partial \bar{p}/\partial z = -g\bar{\rho}$$

$$\mathcal{L} = \left[\frac{1}{2}\rho|\mathbf{u}|^2 - \Phi\rho - \rho U + \lambda\left(p - \bar{p}\right) + \mu\left(g\bar{\rho} + \frac{\partial \bar{p}}{\partial z}\right)\right]$$

Variational calculus yields equation of motion

$$\rho\left(\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \nabla\left(\phi - g\mu\right)\right) = -\left(1-\lambda\right)\nabla\bar{p} - \nabla\left(\rho c^2\lambda\right)$$

A hybrid model

Approach is very extensible, but in the most straightforward case (plane-parallel geometry, constant gravity, inviscid, no heating, single fluid, ideal...) we have a system of equations that looks like:

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \frac{\beta_0}{\rho}\nabla\left(\frac{\gamma\bar{p}\lambda}{\beta_0}\right) = g\frac{\partial\mu}{\partial z}\hat{\mathbf{z}} - \frac{\rho'}{\rho}g\hat{\mathbf{z}}$$

$$\nabla\cdot(\beta_0\mathbf{u}) = \frac{\beta_0}{\bar{p}\gamma}\left(\frac{Q}{\rho T}\left(\frac{\partial\bar{p}}{\partial S}\right)_\rho - \frac{\partial\bar{p}}{\partial t}\right)$$

$$T \equiv U_s(\rho, S) + \frac{\lambda}{\rho}\bar{p}_S(\rho, S)$$

(plus an unmodified continuity equation).

How do we solve these equations numerically?

Testing the hybrid model with MAESTROeX

- Our equations are pretty similar to MAESTROeX[8] (when running under the same assumptions, ideal gas, etc).
- Since Mike's talk was way back on Monday, a reminder that MAESTROeX is...
  - An AMReX code for Low Mach astrophysics in planar and spherical / full-star geometry.
  - Explicit Godunov approach to advection.
  - Multigrid projections to enforce divergence constraints and calculate pressure perturbation.
  - Time-varying, 1D background state held in HSE.
  - Many orders of magnitude more development hours than Toby and I could put into writing a code from scratch!!!
- Can we just modify existing MAESTROeX for the new terms, temperature redefinition, etc? The (unmodified) code will also provide for a good point of comparison.

[8] Fan+ 2019, ArXiv preprint 1908.03634

Testing the hybrid model with MAESTROeX

Hybrid Model

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} + \frac{\beta_0}{\rho}\nabla\left(\frac{\gamma \bar{p} \lambda}{\beta_0}\right) = g\frac{\partial \mu}{\partial z}\hat{\mathbf{z}} + \frac{\rho'}{\rho}\mathbf{g}$$

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MAESTROeX

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} + \frac{\beta_0}{\rho}\nabla\left(\frac{p'}{\beta_0}\right) = \frac{\rho'}{\rho}\mathbf{g}$$

$$\nabla \cdot (\beta_0 \mathbf{u}) = \beta_0\left(S_e - \frac{1}{\overline{\Gamma_1} p_0}\frac{\partial p_0}{\partial t}\right)$$

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MAESTROeX

$$\nabla \cdot (\beta_0 \mathbf{u}) = \beta_0 \left(S_e - \frac{1}{\overline{\Gamma_1} p_0} \frac{\partial p_0}{\partial t} \right)$$

Can manipulate to show that the divergence constraints are essentially the same, bar the modified definition of temperature in the hybrid model:

$$T \equiv U_s(\rho, S) + \frac{\lambda}{\rho} \bar{p}_S(\rho, S)$$

So enforcing the velocity constraint isn't an issue with existing approach.

Testing the hybrid model with MAESTROeX

Hybrid Model

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} + \frac{\beta_0}{\rho}\nabla\left(\frac{\gamma \bar{p} \lambda}{\beta_0}\right) = g\frac{\partial \mu}{\partial z}\hat{\mathbf{z}} + \frac{\rho'}{\rho}\mathbf{g}$$

MAESTROeX

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} + \frac{\beta_0}{\rho}\nabla\left(\frac{p'}{\beta_0}\right) = \frac{\rho'}{\rho}\mathbf{g}$$

β_0 etc are defined in the same way, can define a re-scaled Lagrange multiplier

$$p' = \gamma \bar{p} \lambda$$

The two equations are then essentially the same, bar the μ forcing term. Modifications mainly hinge on coming up for a strategy for evaluating it.

Summary

Hybrid pseudo-incompressible and quasi-hydrostatic models

- Typical sound-proof approximations not valid for all scales.
- Pseudo-incompressible approximation is good...
- ...but larger horizontal scales should instead be treated as quasi-hydrostatic.

Energy-conserving hybrid model

- Typical derivations of sound-proof approximations do not guarantee energy conserving properties of the equation set.
- Hamiltonian approach guides construction of proper model equations.
- Doing so for both pseudo-incompressible and quasi-hydrostatic constraints, we have a new hybrid model to solve.
- Test implementation in MAESTROeX, with proof of energy properties? Watch this space!