Spectra of magnetic energy and secular variation in a dynamo model of Jupiter

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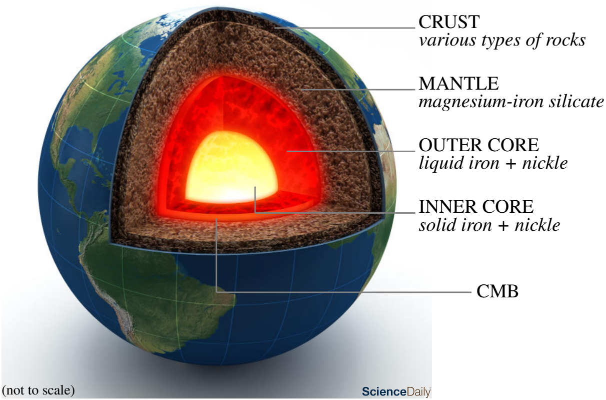
Let's start on Earth . . .



- **core-mantle boundary** (CMB): *sharp boundary* between the non-conducting mantle and the conducting outer core
  $\Rightarrow$ dynamo action entirely confined within the outer core
- **dynamo radius** $r_{\rm dyn}$: top of the dynamo region $\approx r_{\rm cmb}$
- one way to deduce $r_{\rm cmb}$ from *observation at the surface*: magnetic energy spectrum

Gauss coefficients g_{lm} and h_{lm}

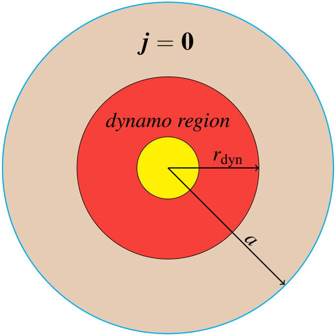
- Outside the dynamo region, $r > r_{\rm dyn}$:

$$\boxed{\boldsymbol{j} = \boldsymbol{0}}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} = \boldsymbol{0} \implies \boldsymbol{B} = -\nabla\Psi$$

$$\nabla \cdot \boldsymbol{B} = 0 \implies \nabla^2\Psi = 0$$

$a =$ *radius of Earth*



- Consider only internal sources,

$$\Psi(r,\theta,\phi) = a \sum_{l=1}^{\infty} \sum_{m=0}^{l} \left(\frac{a}{r}\right)^{l+1} \hat{P}_{lm}(\cos\theta)(g_{lm}\cos m\phi + h_{lm}\sin m\phi)$$

$\hat{P}_{lm}$: *Schmidt's semi-normalised associated Legendre polynomials*

- g_{lm} and h_{lm} can be determined from magnetic field measured at the planetary surface $(r \approx a)$

The Lowes spectrum

- *Average magnetic energy* over a spherical surface of radius r

$$E_B(r) = \frac{1}{2\mu_0}\frac{1}{4\pi}\oint |\boldsymbol{B}(r,\theta,\phi)|^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$$

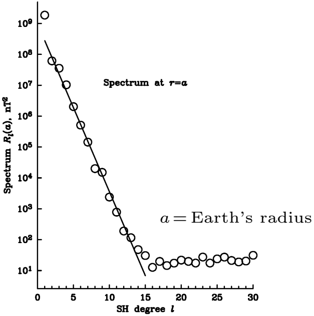
- Inside the source-free region $r_{\mathrm{dyn}} < r < a$,

$$2\mu_0 E_B(r) = \sum_{l=1}^{\infty}\left[\left(\frac{a}{r}\right)^{2l+4}(l+1)\sum_{m=0}^{l}\left(g_{lm}^2 + h_{lm}^2\right)\right]$$

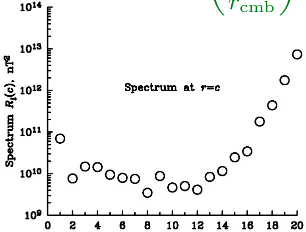
- **Lowes spectrum** (magnetic energy as a function of l):

$$\begin{aligned} R_l(r) &= \left(\frac{a}{r}\right)^{2l+4}(l+1)\sum_{m=0}^{l}\left(g_{lm}^2 + h_{lm}^2\right) \\ &= \left(\frac{a}{r}\right)^{2l+4} R_l(a) \qquad \text{(downward continuation)} \end{aligned}$$

Estimate location of CMB using the Lowes spectrum



$$R_l(r_{\rm cmb}) = \left(\frac{a}{r_{\rm cmb}}\right)^{2l+4} R_l(a)$$



(Robert Parker, UCSD)

- downward continuation from a to $r_{\rm cmb}$ through the mantle ($\boldsymbol{j} = \mathbf{0}$):

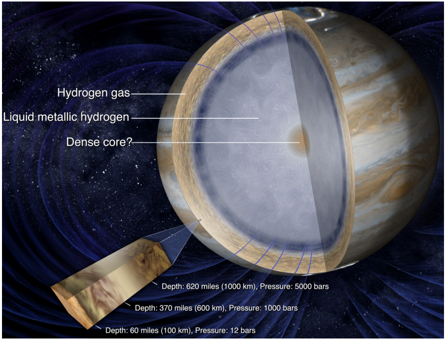
$$\ln R_l(a) = 2\ln\left(\frac{r_{\rm cmb}}{a}\right) l + 4\ln\left(\frac{r_{\rm cmb}}{a}\right) + \ln R_l(r_{\rm cmb})$$

- **white source hypothesis**: turbulence in the core leads to an *even distribution of magnetic energy* across different scales l,

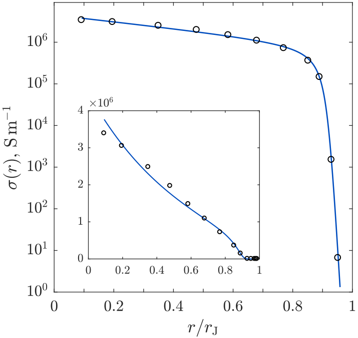
  $R_l(r_{\rm cmb})$ is independent of l

- $r_{\rm cmb} \approx 0.55a \approx 3486\,$km agrees very well with results from seismic waves observations

Interior structure of Jupiter



(NASA JPL)

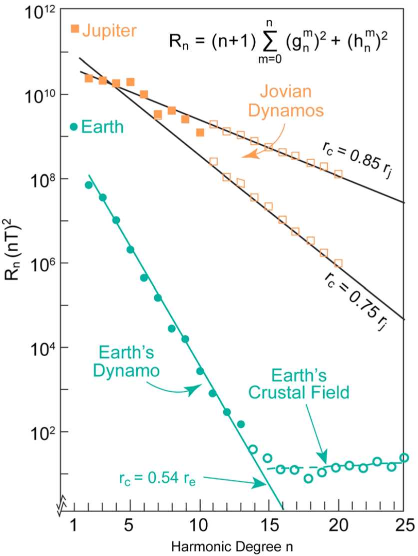


theoretical $\sigma(r)$ (French et al. 2012)

- low temperature and pressure near surface $\Rightarrow$ gaseous molecular H/He
- extremely high temperature and pressure inside $\Rightarrow$ liquid metallic H
- core?
- transition from molecular to metallic hydrogen is continuous
- **conductivity** $\sigma(r)$ varies smoothly with radius r

At what depth does dynamo action start?

Lowes spectrum from the Juno mission



(Connerney et al. 2018)

- Juno's spacecraft reached Jupiter on 4th July, 2016
- currently in a 53-day orbit, measuring Jupiter's magnetic field (and other data)
- $R_l(r_J)$ up to $l = 10$ from latest measurement (8 flybys)
- Lowes' radius: $r_{\text{lowes}} \approx 0.85\, r_{\text{J}}$ ($r_{\text{J}} = 6.9894 \times 10^7 \text{m}$)

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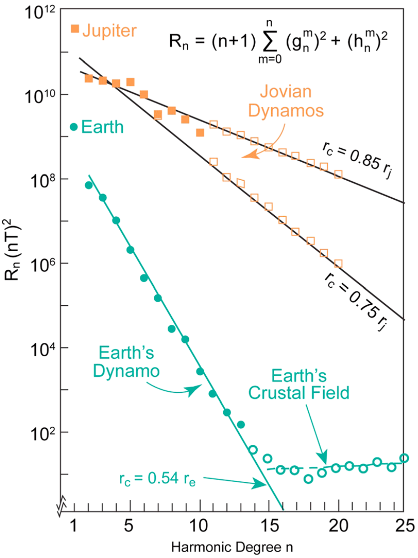

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Questions: with the conductivity profile $\sigma(r)$ varying smoothly,

- meaning of r_{lowes}? $r_{\text{lowes}} = r_{\text{dyn}}$?
- white source hypothesis valid?
- concept of "dynamo radius" r_{dyn} well-defined?

A numerical model of Jupiter

- spherical shell of radius ratio $r_{\text{in}}/r_{\text{out}} = 0.0963$ (small core)
- rotating fluid with electrical conductivity $\sigma(r)$ driven by buoyancy
- convection forced by secular cooling of the planet
- anelastic: linearise about a hydrostatic adiabatic basic state $(\bar{\rho}, \bar{T}, \bar{p}, \dots)$
- dimensionless numbers: Ra, Pm, Ek, Pr

$$\nabla \cdot (\bar{\rho}\boldsymbol{u}) = 0$$

$$\frac{Ek}{Pm}\left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right] + 2\hat{\boldsymbol{z}} \times \boldsymbol{u} = -\nabla\Pi' + \frac{1}{\bar{\rho}}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} - \left(\frac{EkRaPm}{Pr}\right) S \frac{\mathrm{d}\bar{T}}{\mathrm{d}r}\hat{\boldsymbol{r}} + Ek\frac{\boldsymbol{F}_\nu}{\bar{\rho}}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) - \nabla \times (\eta \nabla \times \boldsymbol{B})$$

$$\bar{\rho}\bar{T}\left(\frac{\partial S}{\partial t} + \boldsymbol{u}\cdot\nabla S\right) + \frac{Pm}{Pr}\nabla\cdot\boldsymbol{\mathcal{F}}_Q = \frac{Pr}{RaPm}\left(Q_\nu + \frac{1}{Ek}Q_J\right) + \frac{Pm}{Pr}H_S$$

Boundary conditions: no-slip at r_{in} and stress-free at r_{out}, $S(r_{\text{in}}) = 1$ and $S(r_{\text{out}}) = 0$, electrically insulating outside $r_{\text{in}} < r < r_{\text{out}}$. (Jones 2014)

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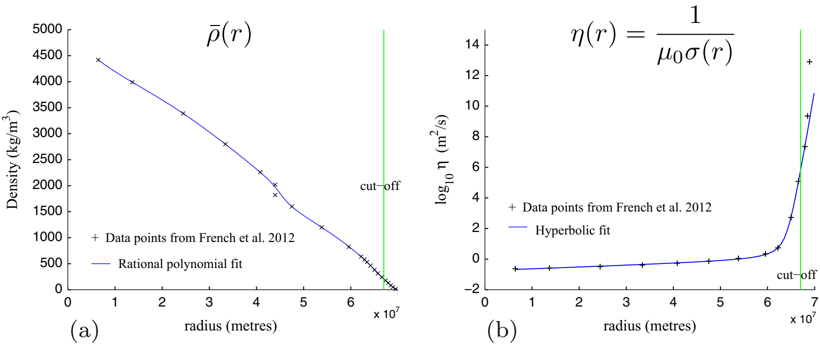
- **a Jupiter basic state:** *C.A. Jones/Icarus 241 (2014) 148–159*

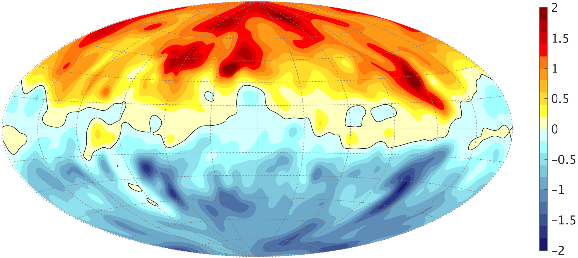


$Ra = 2 \times 10^7,\ Ek = 1.5 \times 10^{-5},\ Pm = 10,\ Pr = 0.1$

radial magnetic field, $B_r(r, \theta, \phi)$

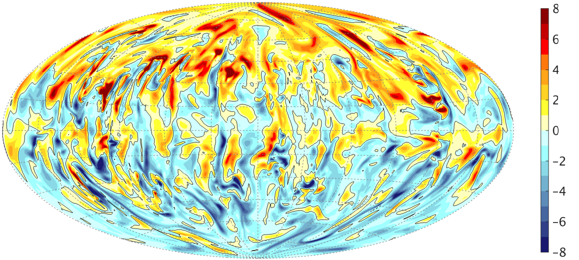
$r = r_{\text{out}}$

dipolar

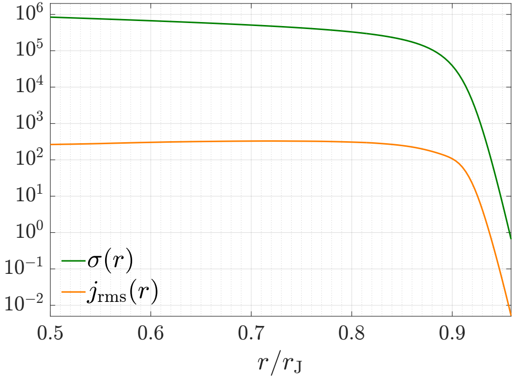


$r = 0.75 r_{\text{out}}$

small scales



Where does the current start flowing?



- average current over a spherical surface of radius r

$$\mu_0 \boldsymbol{j} = \nabla \times \boldsymbol{B}$$

$$j_{\mathrm{rms}}^2(r,t) \equiv \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} |\boldsymbol{j}|^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$$

- j_{rms} drops quickly but smoothly in the transition region, not clear how to define a characteristic “dynamo radius”

Magnetic energy spectrum, $F_l(r)$

- average magnetic energy over a spherical surface:

$$E_B(r) = \frac{1}{2\mu_0}\frac{1}{4\pi}\oint |\boldsymbol{B}(r,\theta,\phi)|^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$$

- Lowes spectrum: recall that if $\boldsymbol{j} = \boldsymbol{0}$, we solve $\nabla^2\Psi = 0$, then

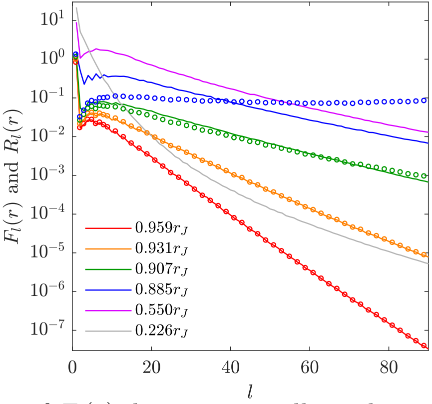
$$2\mu_0 E_B(r) = \sum_{l=1}^{\infty}\left[\left(\frac{a}{r}\right)^{2l+4}(l+1)\sum_{m=0}^{l}\left(g_{lm}^2 + h_{lm}^2\right)\right] = \sum_{l=1}^{\infty} R_l(r)$$

- generally, for the numerical model, $\boldsymbol{B} \sim \sum_{lm} b_{lm}(r) Y_{lm}(\theta,\phi)$,

$$2\mu_0 E_B(r) = \frac{1}{4\pi}\oint |\boldsymbol{B}(r,\theta,\phi)|^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi = \sum_{l=1}^{\infty} F_l(r)$$

$$\boldsymbol{j}(r,\theta,\phi) = \boldsymbol{0} \;\; \text{exactly} \implies R_l(r) = F_l(r)$$

Magnetic energy spectrum at different depth r

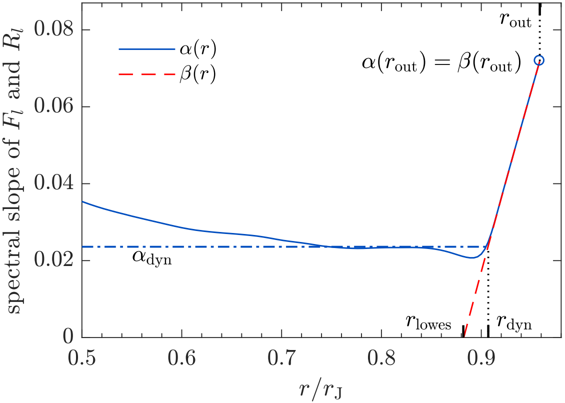


$F_l(r)$: solid lines

$R_l(r)$: circles

- $r > 0.9r_{\mathrm{J}}$: slope of $F_l(r)$ decreases rapidly with r
  $r < 0.9r_{\mathrm{J}}$: $F_l(r)$ maintains the same shape and slope
  $\Rightarrow$ a shift in the dynamics of the system
- $r > 0.9r_{\mathrm{J}}$: $F_l(r) \approx R_l(r)$
  $r < 0.9r_{\mathrm{J}}$: $F_l(r)$ deviates from $R_l(r)$
  $\Rightarrow$ electric current becomes important below $0.9r_{\mathrm{J}}$
- suggests a dynamo radius $r_{\mathrm{dyn}} \approx 0.9r_{\mathrm{J}}$

Spectral slope of $F_l(r)$ and $R_l(r)$



$\log_{10} F_l(r) \sim -\alpha(r) l$

$\log_{10} R_l(r) \sim -\beta(r) l$

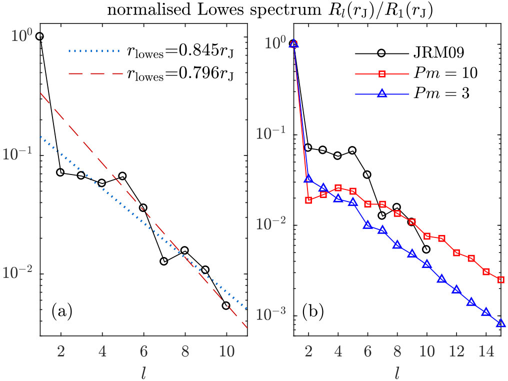
$R_l(r) = \left(\frac{r_\mathrm{out}}{r}\right)^{2l+4} R_l(r_\mathrm{out})$

$\beta(r) = \beta(r_\mathrm{out}) - 2\log_{10}\frac{r_\mathrm{out}}{r}$

- sharp transition in $\alpha(r)$ indicates $r_\mathrm{dyn} = 0.907 r_\mathrm{J}$
- $F_l(r)$ inside dynamo region is not exactly flat ($\alpha_\mathrm{dyn} = 0.024$): white source assumption is only approximate
- r_lowes provides a lower bound to r_dyn: $\beta = 0$ at $r_\mathrm{lowes} = 0.883$

General picture: $\alpha(r_\mathrm{out})$ and α_dyn control r_dyn and r_lowes

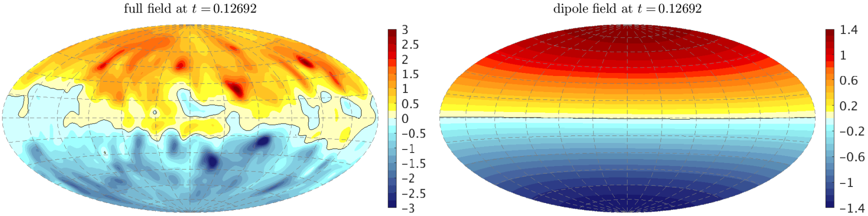
Comparison with Juno data



- noise in Juno data ⇒ results depend on fitting range
- larger Pm gives smaller α_{dyn}, however $\alpha(r_{out})$ also becomes smaller ⇒ r_{dyn} remains roughly the same
- $R_l(r_J)$ is shallower in the numerical model than from Juno observation
  - the metallic hydrogen layer could be deeper than predicted by theoretical calculation
  - the existence of a stably stratified layer below the molecular layer
  - our numerical model has more small-scale forcing than Jupiter does

Time variation of B_r at the surface

Pm=10

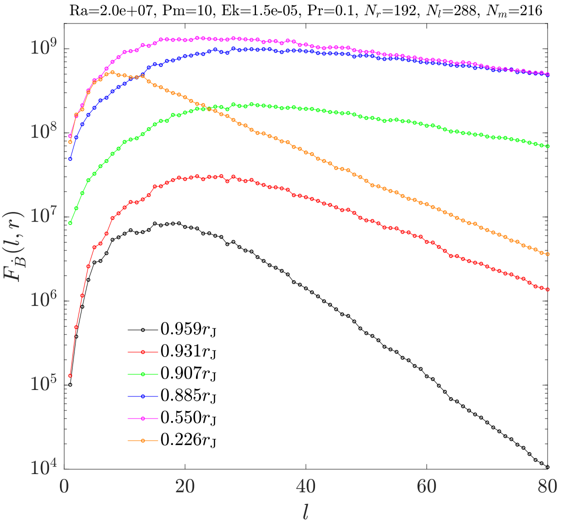


- dipole is slowly varying compared to other modes
- secular variation: $\dot{\boldsymbol{B}} = \dfrac{\partial \boldsymbol{B}}{\partial t}$
- for $\boldsymbol{j} = \boldsymbol{0}$, Lowes spectrum for secular variation:

$$R_{\dot{B}}(l, r) = \left(\frac{a}{r}\right)^{2l+4} (l+1) \sum_{m=0}^{l} \left(\dot{g}_{lm}^2 + \dot{h}_{lm}^2\right)$$

Secular variation spectrum $F_{\dot{B}}(l,r)$

$$\sum_{l=1}^{\infty} F_{\dot{B}}(l,r) = \frac{1}{4\pi} \oint |\dot{\boldsymbol{B}}(r,\theta,\phi)|^2 \sin\theta \, d\theta \, d\phi$$



geomagnetic SV

surface (Holme & Olsen 2006)

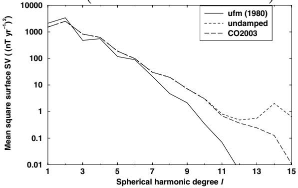


Figure 1. Spectra of *SV* models at Earth's surface.

CMB (Holme et al. 2011)

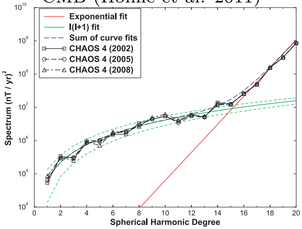


Figure 2. Spectra of the CHAOS-4 SV at the CMB, $r = c$. Green line gives theoretical model, dashed lines approximate 1σ error bounds.

A spectral correlation time τ_l

- a correlation time for different mode l:

$$\tau_l(r) = \left\langle \sqrt{\frac{F(l,r)}{F_{\dot{B}}(l,r)}} \right\rangle_t$$

- for $\boldsymbol{j} = \mathbf{0}$, τ_l becomes independent of r:

$$\tau_l = \left\langle \sqrt{\frac{\sum_{m=0}^{l} \left(g_{lm}^2 + h_{lm}^2\right)}{\sum_{m=0}^{l} \left(\dot{g}_{lm}^2 + \dot{h}_{lm}^2\right)}} \right\rangle_t$$

- two-parameter power law (e.g. Holme & Olsen 2006):

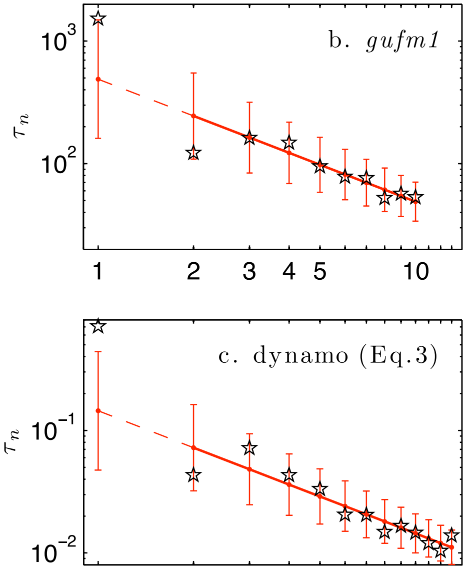
$$\tau_l = \tau_{\mathrm{SV}} \cdot l^{-\gamma}, \quad 1.32 < \gamma < 1.45$$

  $\tau_{\mathrm{SV}} \sim$ a secular variation time scale

- one-parameter power law (e.g. Christensen & Tilgner 2004):

$$\tau_l = \tau_{\mathrm{SV}} \cdot l^{-1}$$

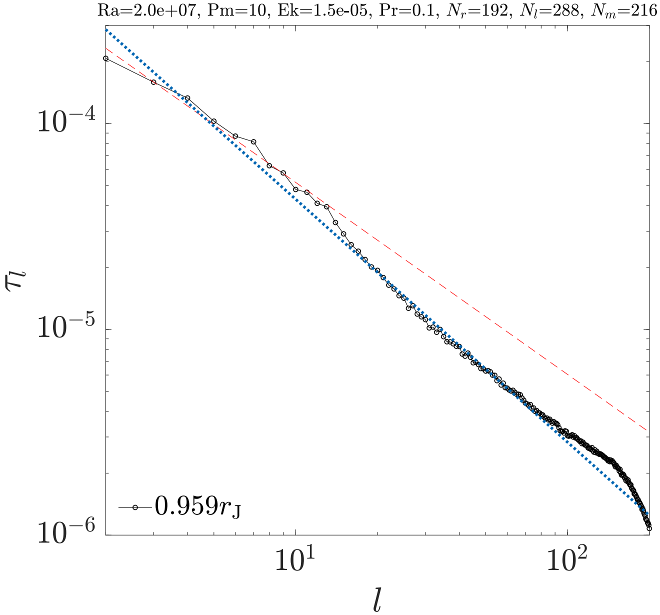
Example of τ_l in geodynamo



(Lhuillier et al. 2011)

$$\tau_l = \tau_{\mathrm{SV}} \cdot l^{-1}$$

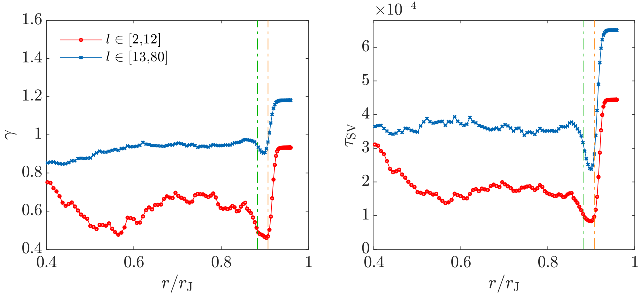
Spectral correlation time τ_l in Jupiter dynamo model



γ and τ_{SV} in Jupiter dynamo model

$$\tau_l = \tau_{\mathrm{SV}} \cdot l^{-\gamma}$$

$\underline{Pm = 10}$



$\underline{Pm = 3}$