Helly graphs and groups

Young Geometric Group Theory X

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Sources

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Helly groups

Examples of groups acting geometrically on Helly graphs:
(Gromov) hyperbolic groups, (cocompact) $\mathrm{CAT}(0)$ cubical groups, uniform lattices in many Euclidean buildings, FC-type Artin groups, finite-type Garside groups, fin. pres. graphical C(4)-T(4) small cancellation groups,. . .

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Properties of Helly groups:
biautomaticity, finiteness properties, finitely many conjugacy classes of finite subgroups, Farrell-Jones conjecture, coarse Baum-Connes conjecture, EZ-boundary, . . .

Outline of the course:

1. Helly property, injective metric spaces, Helly graphs
2. Features of Helly graphs
3. Helly groups: examples and properties

Helly property

Definition (Helly property)

A family of subsets of a set has a *(finite)* *Helly property* if every (finite) subfamily of pairwise intersecting subsets has a nonempty intersection.

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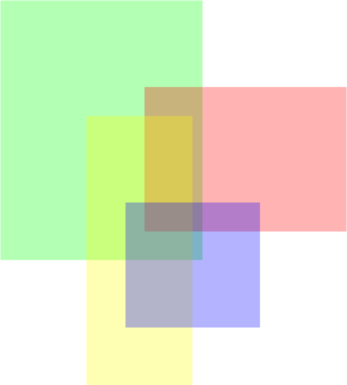
Example (Helly families)

1. axis-parallel boxes in $\mathbb{R}^n$
2. finite subtrees of a tree
3. a finite family of half-spaces of a $\mathrm{CAT}(0)$ cube complex.



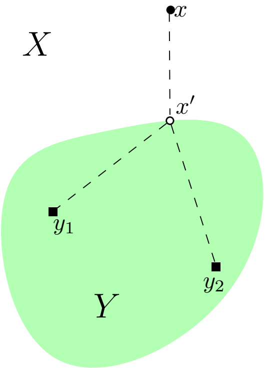
Helly property - examples

Example (Gated subsets)

A subset Y of a metric space (X, d) is *gated* if for every point $x \in X$ there exists a vertex $x' \in Y$, called the *gate* of x, such that $x' \in I(x, y)$, for every $y \in Y$. A finite family of gated subsets has the Helly property.

Example (Intervals in lattices)

A *lattice* is a poset $(P, \leqslant)$ with g.l.b. (called *meet*) and l.u.b. (*join*) for each pair of elements. An *interval* in a lattice is a subset of the form $\{x | a \leqslant x \leqslant b\}$. A finite family of intervals in a lattice has the Helly property.



Injective metric spaces

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Example

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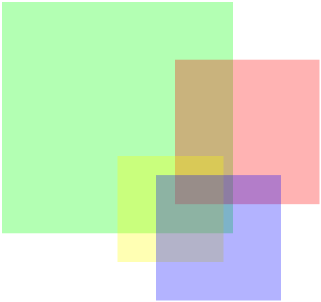
2. an $\mathbb{R}$-tree
3. $(\mathbb{R}^2, d_2)$ is not injective!



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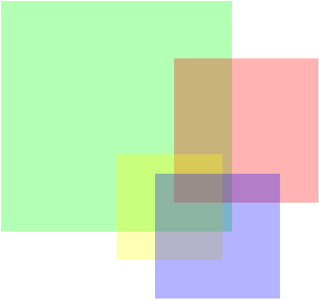


Remark

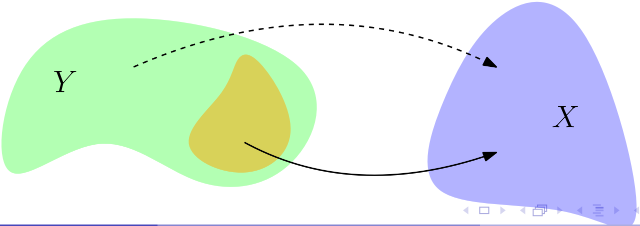
Actually, the definition of an injective space above is not so proper...

Injective spaces

Theorem (Characterizations of injectivity)

Let (X, d) be a geodesic metric space. TFAE:

1. *X is injective*
2. *X is hyperconvex*
3. *[Aronszajn-Panitchpakdi, 1956] (Y, X) has the extension property, for every metric space Y (for the category of metric spaces with 1-Lipschitz maps)*
4. *X is an absolute retract (for the category of metric spaces with 1-Lipschitz maps)*



Injective hull

Definition (Injective hull)

An *injective hull* (or *tight span*, or *injective envelope*, or *hyperconvex hull*) of (X, d) is a pair $(e, E(X))$ where $e\colon X \to E(X)$ is an isometric embedding into an injective metric space $E(X)$, and such that no injective proper subspace of $E(X)$ contains $e(X)$. Two injective hulls $e\colon X \to E(X)$ and $f\colon X \to E'(X)$ are *equivalent* if they are related by an isometry $i\colon E(X) \to E'(X)$.

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Remark

Injective hulls were rediscovered by Dress in 1984, Chrobak-Larmore in 1994...

Isbell's construction

Let (X, d) be a metric space. Consider the space $\mathbb{R}^X$ of real-valued functions with the supremum metric $d(f, g) = \sup_{x \in X} |f(x) - g(x)|$.

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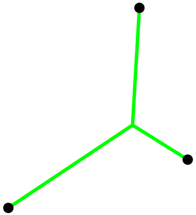
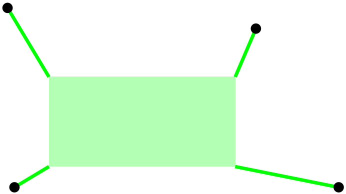
$(e, E(X))$ is the injective hull of X.

Injective hulls - examples





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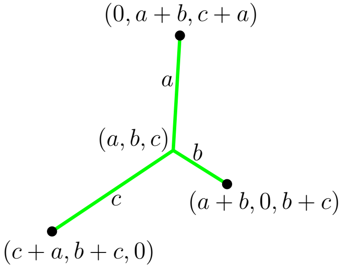

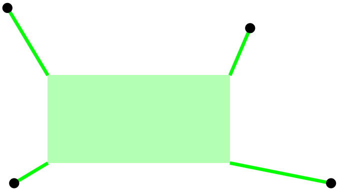

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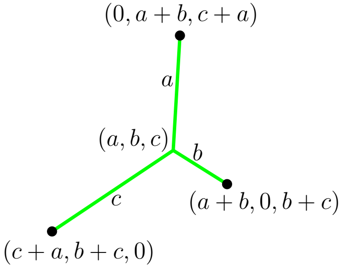
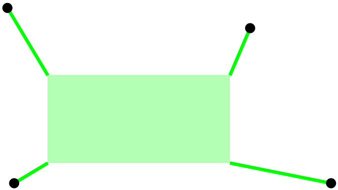
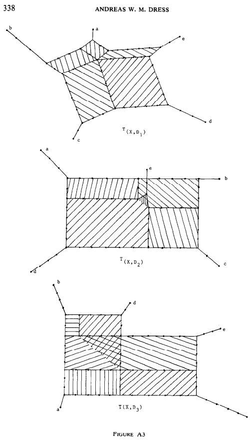






Geodesic bicombing

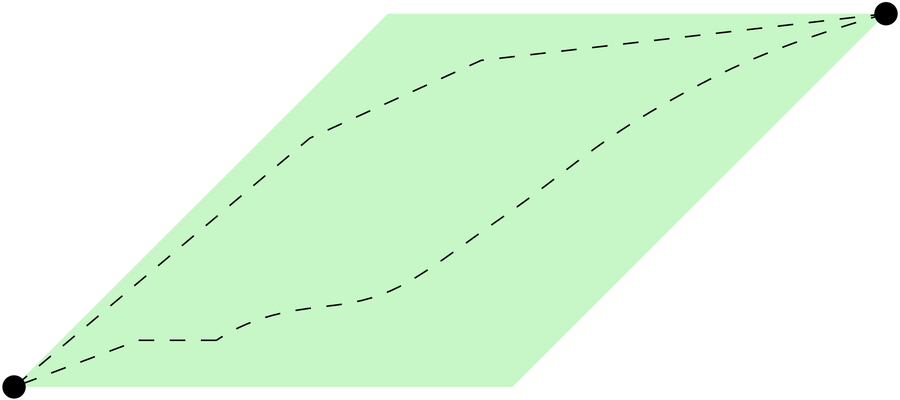
Definition (Geodesic bicombing)

A *geodesic bicombing* on a metric space (X, d) is a map

$$\sigma\colon X \times X \times [0,1] \to X,$$

such that for every pair $(x, y) \in X \times X$ the function $\sigma_{xy} := \sigma(x, y, \cdot)$ is a constant speed geodesic from x to y. We call σ *convex* if the function $t \mapsto d(\sigma_{xy}(t), \sigma_{x'y'}(t))$ is convex for all $x, y, x', y' \in X$. The bicombing σ is *consistent* if $\sigma_{pq}(\lambda) = \sigma_{xy}((1-\lambda)s + \lambda t)$, for all $x, y \in X$, $0 \le s \le t \le 1$, $p := \sigma_{xy}(s)$, $q := \sigma_{xy}(t)$, and $\lambda \in [0,1]$. It is called *reversible* if $\sigma_{xy}(t) = \sigma_{yx}(1-t)$ for all $x, y \in X$ and $t \in [0,1]$.

Bicombing in $(\mathbb{R}^2, d_\infty)$



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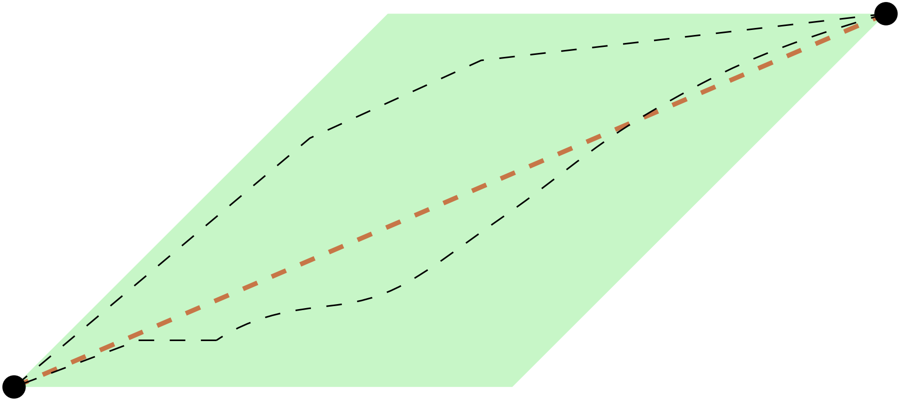

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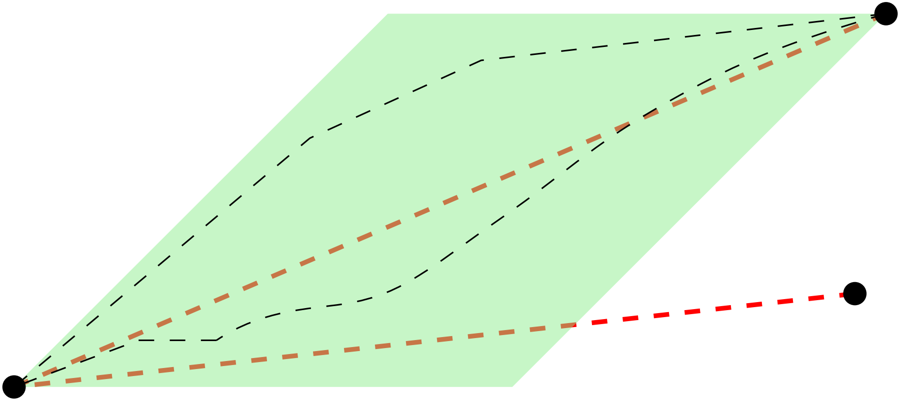

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Remark

In particular, the bicombing above is invariant under automorphisms.

Properties of injective metric spaces

1. contractibility
2. fixed point properties for finite group actions
3. classification of isometries
4. Flat Torus theorem [Descombes-Lang]
5. characterization of hyperbolicity via non-existence of flats

...

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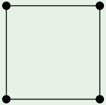
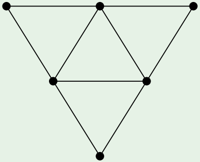


clique-Helly not Helly



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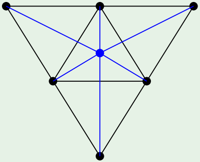
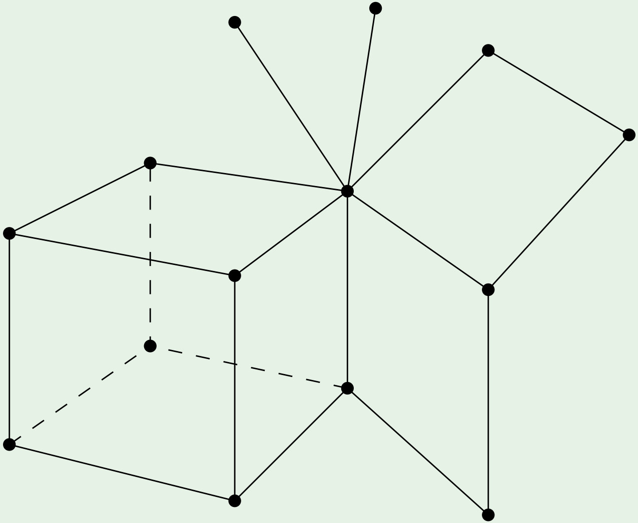


Helly

Helly graphs - example

Example (Thickening of a $\mathrm{CAT}(0)$ cube complex)



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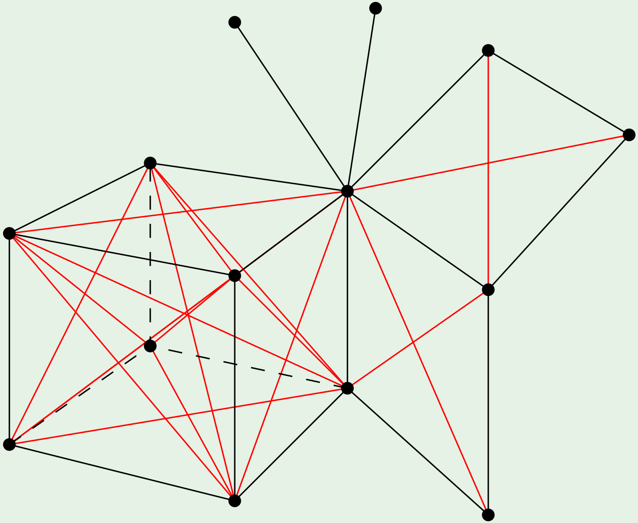

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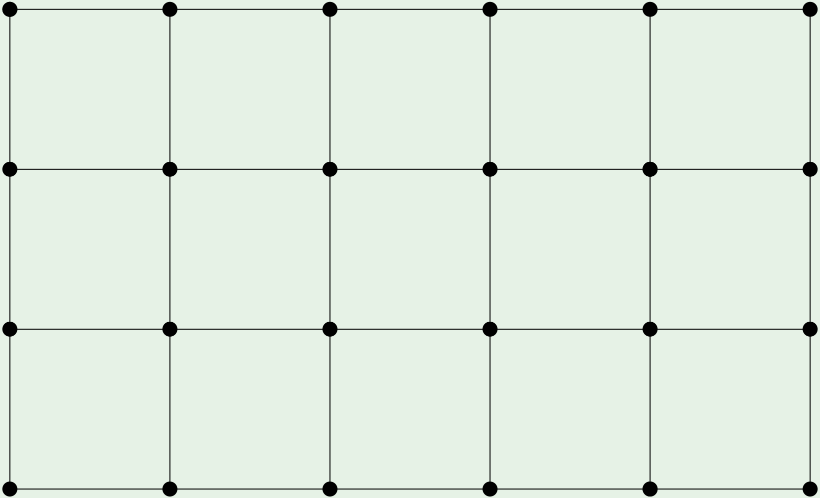

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