Helly graphs and groups

Young Geometric Group Theory X

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Helly graphs

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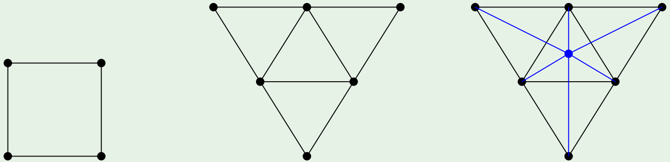
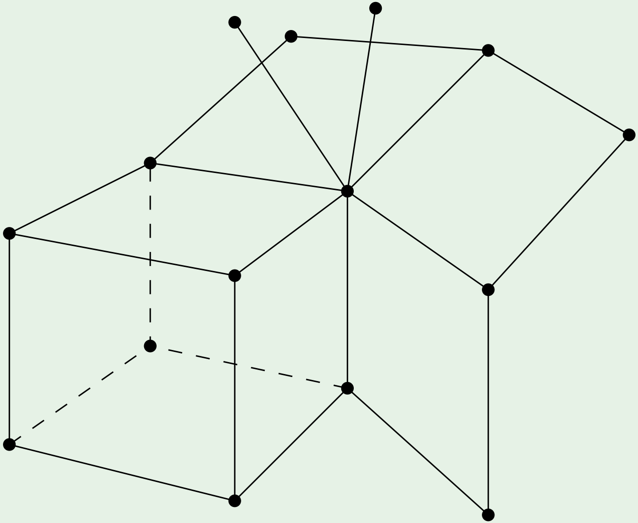


clique-Helly not Helly   not clique-Helly   Helly

Helly graphs - example

Example (Thickening of a $\mathrm{CAT}(0)$ cube complex)



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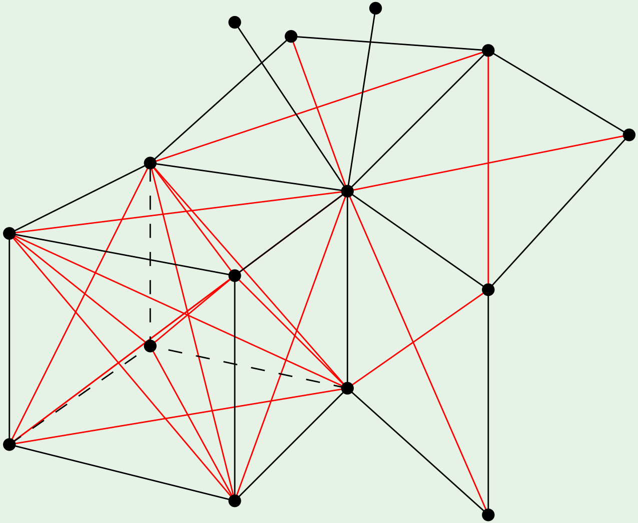

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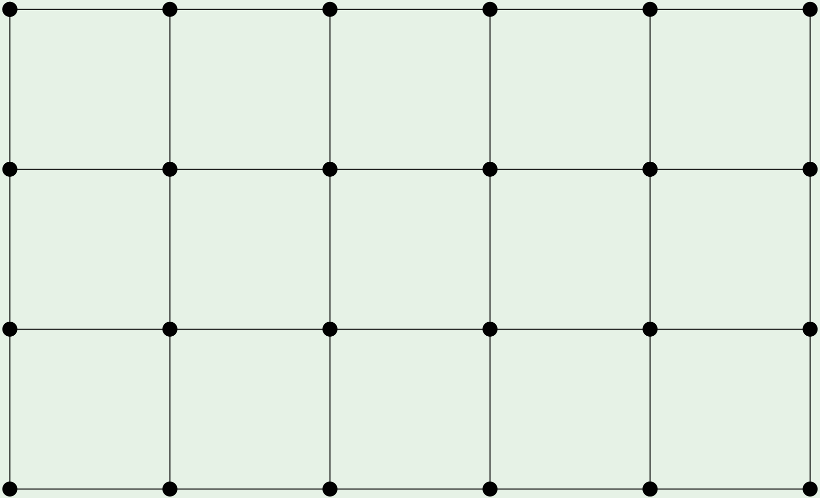

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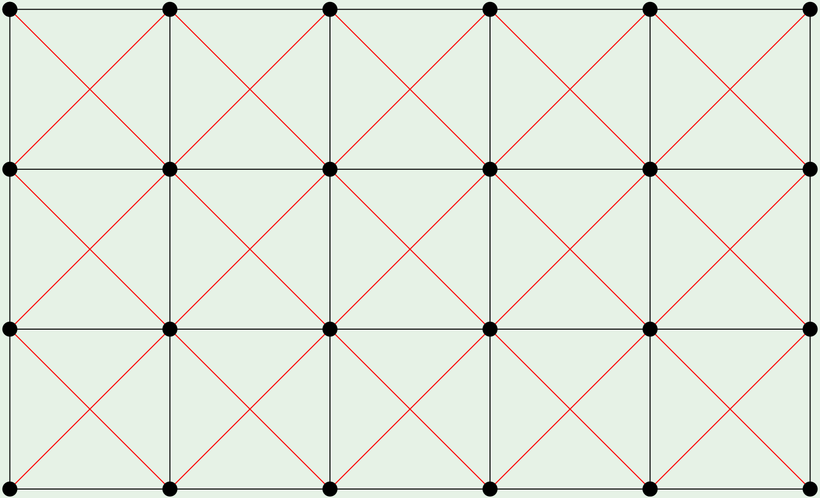

Helly graph

Definition (Weakly modular graph)

A graph is *weakly modular* if, for every three vertices u, v, w with $n := d(u, v) = d(u, w)$ the following two conditions are satisfied:

- (T) if $v \sim w$ then there exists $x \sim u, v$ with $d(u, x) = n - 1$
- (Q) if there exists $z \sim v, w$ with $d(u, z) = n + 1$ then there exists $x \sim u, v$ with $d(u, x) = n - 1$

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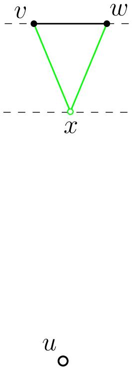

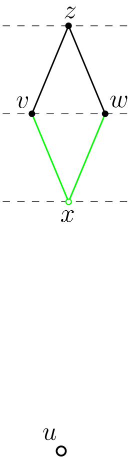

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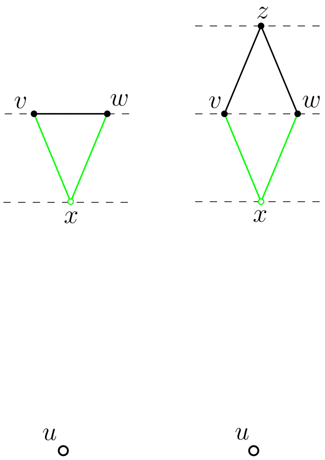

Theorem

Helly graphs are weakly modular.

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Helly graphs are weakly modular. Moreover, they satisfy a stronger version of (Q):

- *(Q') if there exists* $z \sim v, w$ *with* $d(u, z) = n + 1$ *then there exists* $x \sim u, v$ *with* $d(u, x) = n - 1$*, and* $x' \sim z, v, w, x$

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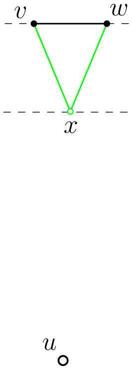

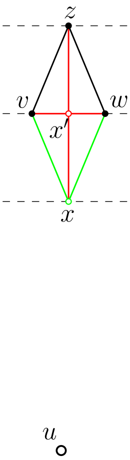

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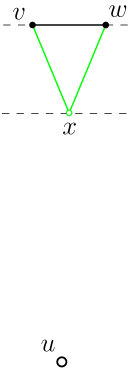

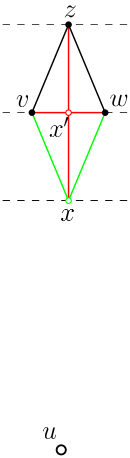

Corollary

The triangle complex of a Helly graph is simply connected. The isoperimetric function is at most quadratic.

Dismantlability

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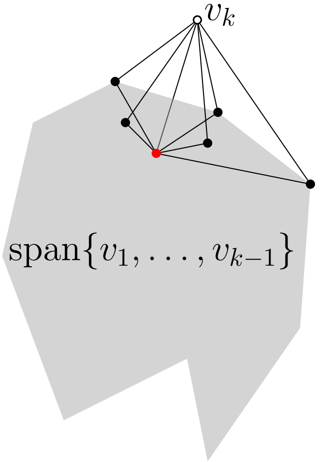
A finite graph Γ is *dismantlable* if its vertices can be enumerated as $v_1, v_2, v_3, ..., v_n$ such that for every $1 < k \leqslant n$ the vertex v_k is *dominated* in the subgraph induced by $v_1, \ldots, v_k$.



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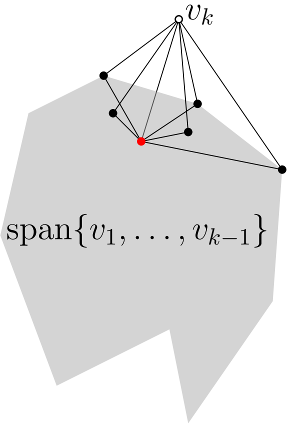
Corollary

The clique complex $X(\Gamma)$ of a locally finite Helly graph Γ is contractible. Finite groups acting on such Helly graphs fix cliques. Fixed point sets are contractible.



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Theorem (Characterizations of Helly graphs)

For a locally finite graph Γ TFEA:

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Theorem (Local-to-global characterization)

A graph is Helly iff it is clique-Helly and its triangle complex is simply connected.

Proof of the local-to-global characterization

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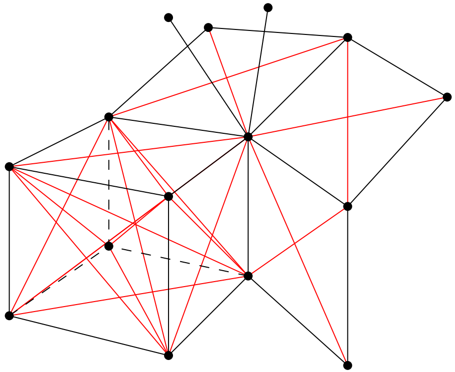
Example

Cocompact $\mathrm{CAT}(0)$ cubical groups are Helly.
The proof goes via convexity of balls or via the local-to-global characterization.



Finite-type Artin groups

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A finite simplicial graph Γ with edges labelled by $\{2, 3, 4, \ldots\}$ defines a presentation of the *Artin group* A_Γ:

$$A_\Gamma = \langle a \in V(\Gamma) \mid \underbrace{aba\cdots}_{m} = \underbrace{bab\cdots}_{m} \text{ for each edge } ab \text{ labelled with } m \rangle$$

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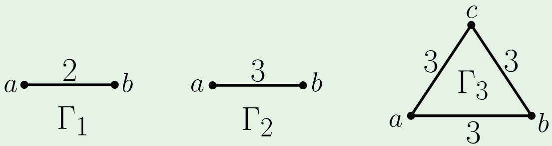

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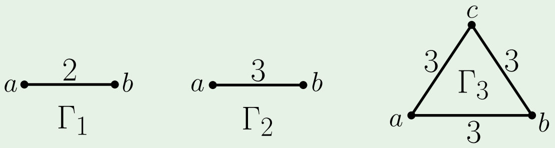

$$A_{\Gamma_1} = \langle a, b \mid ab = ba\rangle \cong \mathbb{Z}^2;\ A_{\Gamma_2} = \langle a, b \mid aba = bab\rangle$$
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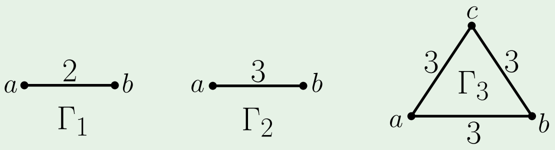

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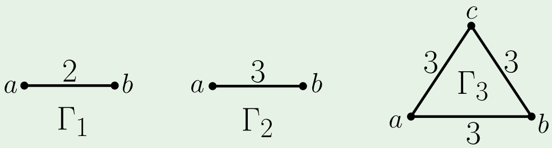

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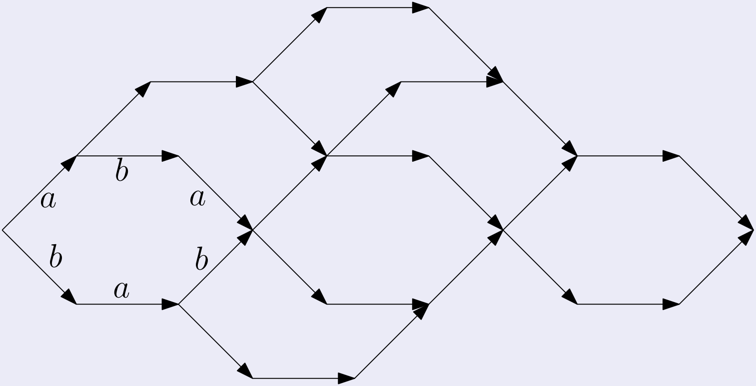
Consider a 'thickening' of the Cayley complex:



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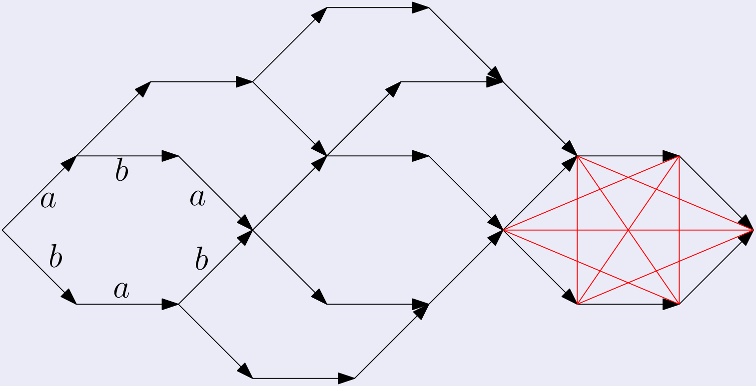

Garside groups

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C(4)-T(4) small cancellation groups

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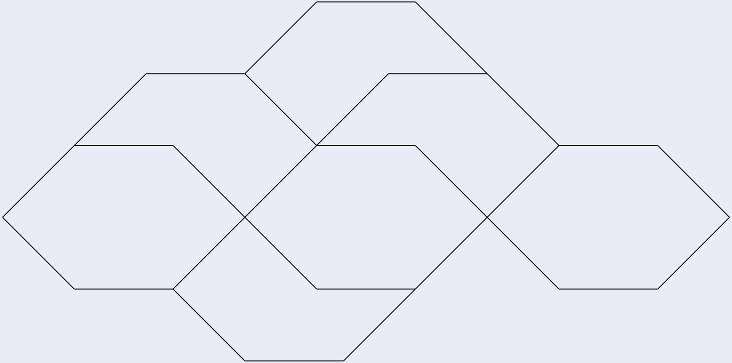

Buildings

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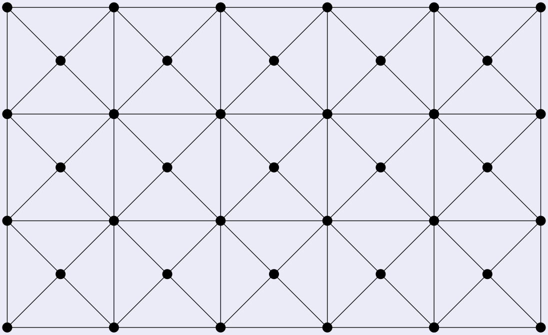

Properties of Helly groups

Theorem (Properties of Helly groups)

Let G be a group acting geometrically on a Helly graph Γ. Then:

1. *The clique complex $X(\Gamma)$ of Γ is a finite-dimensional cocompact model for the classifying space $\underline{E}G$ for proper actions. As a particular case, G is always of type F_∞ and of type F when it is torsion-free.*

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4. *G has at most quadratic Dehn function.*

Constructions preserving Hellyness

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Let $G, G_1, G_2, \ldots, G_n$ be Helly groups. Then:

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Proof.

"Amalgamation of Helly graphs along a vertex is Helly. Strong product of Helly graphs is Helly. Fixed-point set is Helly." □

Hellyfication

Theorem (Jawhari-Pouzet-Misane, Pesch)

For every graph Γ *there exists a minimal Helly graph* $Helly(\Gamma)$, *called Hellyfication of* Γ *into which* Γ *embeds isometrically.*

Proof.

Consider the space $\mathbb{Z}^{\Gamma}$ of integer-valued functions with the supremum metric $d(f,g) = \sup_{x \in \Gamma} |f(x) - g(x)|$.

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Additionaly, Helly$(\Gamma) = E(\Gamma) \cap \mathbb{Z}^\Gamma$.

□

Injective hull vs Hellyfication

Theorem

Let Γ be a locally finite Helly graph.

1. *The injective hull $E(\Gamma)$ of Γ is proper and has the structure of a locally finite polyhedral complex with only finitely many isometry types of n-cells, isometric to injective polytopes in $(\mathbb{R}^n, d_\infty)$, for every $n \geq 1$. Moreover, $d_H(E(\Gamma), e(\Gamma)) \leq 1$. Furthermore, if Γ has uniformly bounded degrees, then $E(\Gamma)$ has finite combinatorial dimension.*

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Corollary

Helly groups act geometrically on spaces with convex, reversible, consistent geodesic bicombing = act geometrically on $\mathrm{CAT}(0)$ -like spaces

Further properties of Helly groups

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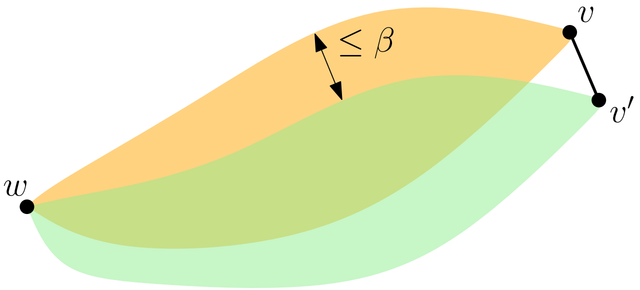
Proof.

Follows immediately from results by Descombes-Lang, Kasprowski-Rüping, Fukaya-Oguni. □

β-stable intervals

Definition (Lang)

For $\beta \geq 1$, the graph Γ has *β-stable intervals* if for every triple of vertices w, v, v' with $v \sim v'$, we have $d_H(I(w,v), I(w,v')) \leq \beta$, where d_H denotes the Hausdorff distance.



Remark

This property is equivalent to the FFTP.

Graphs with β-stable intervals

Theorem (Lang)

Let Γ be a locally finite graph with β-stable intervals. Then the injective hull of Γ is proper (that is, bounded closed subsets are compact) and has the structure of a locally finite polyhedral complex with only finitely many isometry types of n-cells, isometric to injective polytopes in $(\mathbb{R}^n, d_\infty)$, for every $n \geq 1$.

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Theorem

Weakly modular graphs (in particualar, Helly graphs) have 1-stable intervals.