Helly graphs and groups

Young Geometric Group Theory X

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Injective hull vs Hellyfication

Theorem

Let Γ be a locally finite Helly graph.

1. *The injective hull $E(\Gamma)$ of Γ is proper and has the structure of a locally finite polyhedral complex with only finitely many isometry types of n-cells, isometric to injective polytopes in $(\mathbb{R}^n, d_\infty)$, for every $n \geq 1$. Moreover, $d_H(E(\Gamma), e(\Gamma)) \leq 1$. Furthermore, if Γ has uniformly bounded degrees, then $E(\Gamma)$ has finite combinatorial dimension.*
2. *A group acting cocompactly, properly or geometrically on Γ acts, respectively, cocompactly, properly or geometrically on its injective hull $E(\Gamma)$.*

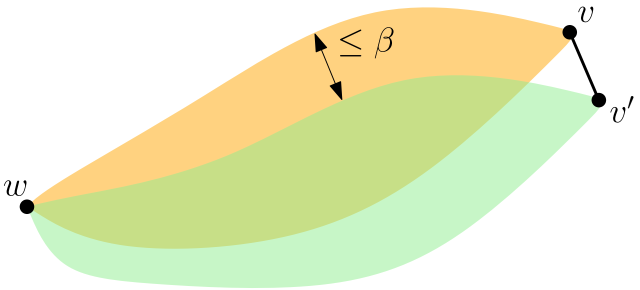
Corollary

Helly groups act geometrically on spaces with convex, reversible, consistent geodesic bicombing
$=$ act geometrically on $\mathrm{CAT}(0)$ -like spaces

β-stable intervals

Definition (Lang)

For $\beta \geq 1$, the graph Γ has *β-stable intervals* if for every triple of vertices w, v, v' with $v \sim v'$, we have $d_H(I(w,v), I(w,v')) \leq \beta$, where d_H denotes the Hausdorff distance.



Remark

This property is equivalent to the FFTP.

Graphs with β-stable intervals

Theorem (Lang)

Let Γ be a locally finite graph with β-stable intervals. Then the injective hull of Γ is proper (that is, bounded closed subsets are compact) and has the structure of a locally finite polyhedral complex with only finitely many isometry types of n-cells, isometric to injective polytopes in $(\mathbb{R}^n, d_\infty)$, for every $n \geq 1$.

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Example

For Γ being the 1-skeleton of a regular cubical grid in $\mathbb{E}^3$ or a regular triangulation of $\mathbb{E}^2$ we have $d_H(e(\Gamma), E(\Gamma)) = \infty$, equivalently, $d_H(e(\Gamma), \mathrm{Helly}(\Gamma)) = \infty$.

Coarse Helly

Definition

A metric space (X, d) has the *coarse Helly property* if there exists $\delta \geq 0$ such that for any family $\{B_{r_i}(x_i) : i \in I\}$ of pairwise intersecting closed balls of X, the intersection $\bigcap_{i \in I} B_{r_i+\delta}(x_i)$ is not empty.

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$f(x)+f(y) > d(x, z)+\delta+\epsilon+d(y, z)-\delta = d(x, z)+d(y, z)+\epsilon \geq d(x, y)+\epsilon$, a contradiction. □

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Theorem (Haettel-Hoda-Petyt)

Hierarchically hyperbolic groups (in particular mapping class groups) act metrically properly and cocompactly on coarse Helly spaces. In particular, they are semihyperbolic.

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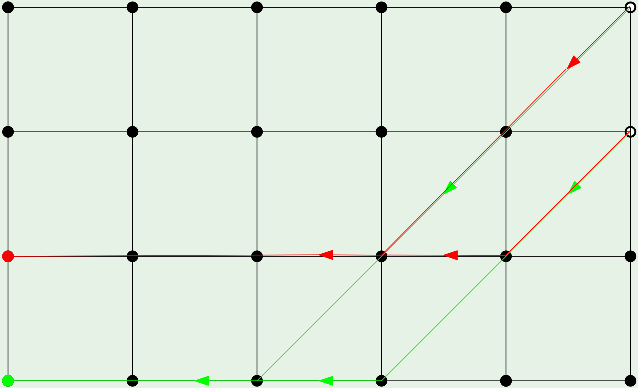
Then G is biautomatic.

Biautomaticity

Example (Biautomaticity of $\mathrm{CAT}(0)$ cubical groups)



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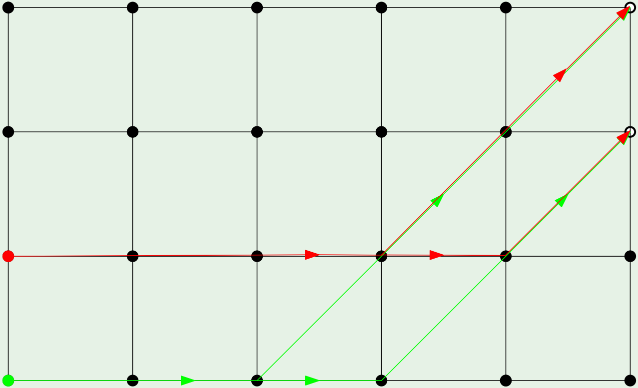

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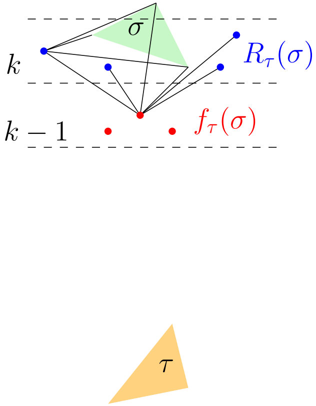

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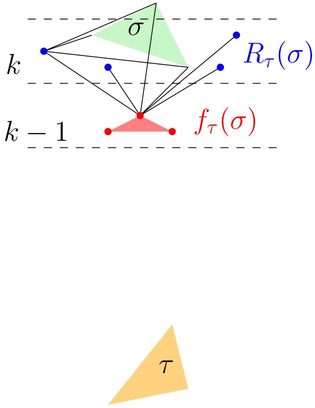
$f_\tau(\sigma)$ is called the *imprint* of σ with respect to τ. It is a clique.



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A sequence of cliques $(\sigma_0, \sigma_1, \ldots, \sigma_k)$ of a Helly graph Γ is called a *normal clique-path* if the following local conditions hold:

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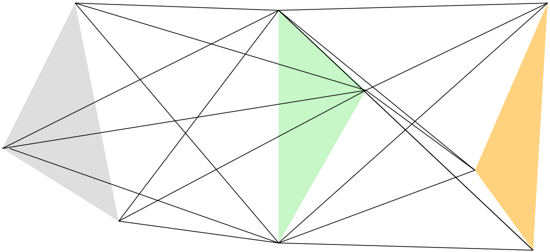

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For any pair τ, σ of cliques of a Helly graph Γ at uniform distance k, there exists a unique normal clique-path $\gamma_{\tau\sigma} = (\tau = \sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_k = \sigma)$, whose cliques are given by

$$\sigma_i = f_\tau(\sigma_{i+1}) \text{ for each } i = k-1, \ldots, 2, 1,$$

and any sequence of vertices $P = (s_0, s_1, \ldots, s_k)$ such that $s_i \in \sigma_i$ for $0 \leq i \leq k$ is a shortest path from s_0 to s_k. In particular, any two vertices p, q of G are connected by a unique normal clique-path γ_{pq}.

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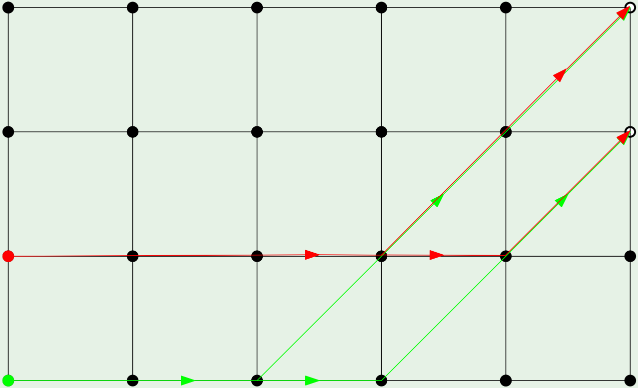
...and check it is as required.

Normal clique path

Example (Normal clique path)



Few other results

Theorem (Hoda)

A crystallogrphic group is Helly iff it is cubical.

Theorem (CCHO, Hirai, Haettel)

Lattices in many (extended) buildings are Helly.

Theorem (Haettel)

Let $\mathbb{K}$ be a local field (with characteristic different from 2 if $\mathbb{K}$ in non-Archimedean), and let $n \geq 3$. Then $SL(n, \mathbb{K})$ does not act properly and coboundedly on an injective metric space.

Open questions

1. Find more examples of groups acting nicely on (coarse) Helly spaces/injective spaces.
   E.g.: What about Artin groups, lattices in buildings, ...?
2. Which $\mathrm{CAT}(0)$ properties hold for Helly groups/groups acting nicely on Helly graphs/injective spaces?
3. When an amalgam of Helly groups is Helly?
4. Study the combinatorial dimension of groups/spaces!
5. Study the coarse geometry of Helly graphs and injective metric spaces.