Equivariant Cayley Complex Embeddings

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Two families, both alike in behaviour,
In fair $\mathbb{S}^3$, where we lay our scene...

Dramatis Personae

Finite groups which admit a topological, faithful action over $\mathbb{S}^3$.

Finite groups which have a Cayley complex H which embeds equivariantly in $\mathbb{S}^3$.

Question: How do these families relate to each other?

Interlude

The word "equivariant" describes an embedding f of a Cayley complex H in a manifold M for which this diagram commutes:

$$\begin{array}{ccc} H & \xrightarrow{f} & M \\ \gamma_g \Big\downarrow & & \phi_g \Big\downarrow \\ H & \xrightarrow{f} & M \end{array}$$

Forward Implication

WARWICK

Theorem

Every finite group which admits a faithful topological action has a Cayley complex which embeds equivariantly in $\mathbb{S}^3$.

How so?

- Let G be such a group.
- By Pardon's Theorem and the Spherical Space Form Theorem, G has a faithful isometric action over $\mathbb{S}^3$.
- We construct an equivariant embedding, with respect to appropriate actions of G, of a simply connected 2-complex.
- We prove and use a version of Babai's Theorem for 2-complexes to produce a Cayley complex of G equivariantly embedded in $\mathbb{S}^3$.

Backward Implication

WARWICK

Theorem

Every finite group which has a Cayley complex which embeds in $\mathbb{S}^3$ admits a faithful topological action in $\mathbb{S}^3$ for which this embedding is equivariant.

How so?

Theorem (Georgakopoulos, Kim)

Every Whitney 2-complex has a unique embedding in $\mathbb{S}^3$ up to homeomorphism.

- Let $f : H \to \mathbb{S}^3$ be an embedding of a Cayley complex of G.
- We construct an embedding $f' : H' \to \mathbb{S}^3$ and an action $\gamma' : G \times H' \to H'$ such that $H \subseteq H'$, $f'|_H = f$, $\gamma'|_H = \gamma$, and H' is Whitney.
- We apply the Georgakopoulos-Kim Theorem.
- We are done!

Generalizations

Question: Does the forward implication hold for $\mathbb{R}^3$?

Answer: Still open.

QuestionL Does the backward implication hold for $\mathbb{R}^3$?

Answer: No! $\mathbb{Z}_3 \times \mathbb{Z}_3$ offers a counter-example.

Question: How do these results extend to infinite groups?


The End